stochastic
processes
and their

Large deviations for a stochastic Volterra-type equation in the Besov–Orlicz space

Boualem Djehiche[a, *], M'hamed Eddahbi[b]

[a] Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden

[b] Département de Mathématiques et d'Informatique Faculté des Sciences et Techniques, Université Cadi Ayyad, B.P. 618, Marrakech, Maroc, Morocco

Received 14 January 1998; received in revised form 30 September 1998; accepted 13 November 1998

Abstract

In this paper, we investigate the regularity of the solutions of a class of two-parameter Stochastic Volterra-type equations in the anisotropic Besov–Orlicz space $\mathscr{B}^0_{\tau,\omega}$ modulated by the Young function $\tau(t) = \exp(t^2) - 1$ and the modulus of continuity $\omega(t) = (t(1 + \log(1/t)))^{1/2}$. Moreover, we derive in the Besov–Orlicz norm a large deviation estimate of Freidlin–Wentzell type for the solution.

1991 Mathematics Subject Classification: Primary 92D30; secondary 60J27; 60F17

Keywords: Brownian sheet; Besov–Orlicz norm; Hyperbolic stochastic partial differential equation; Large deviations; Volterra equation

1. Introduction and results

Recently, there has been a growing interest in the study of stochastic partial differential equations (SPDEs) of hyperbolic type due to their importance in many applied areas. Our primary ambition is to study statistical inference for the solutions of this type of equations, taking advantage of wavelet bases which are known to enjoy good properties for statistical estimation – especially when the underlying function spaces allow less regular functions – combined with the Laplace–Varadhan Principle which yields bounds on the rate of performance of any consistent parameter estimator. It is then natural to seek a class of function spaces that produces a stronger topology than the ones induced by, e.g. the sup-norm and the Hölder norm, thus gives more (semi-)continuous functions which yields of course sharper large deviations estimates and gives a broader scope to the Laplace–Varadhan Principle. Since the driving noise in such equations is the white noise, it is natural to consider a class of anisotropic Besov–Orlicz spaces of L^p-smooth functions that enjoys the above-mentioned properties and includes (almost surely) the driving white noise. To this end, the purpose of

* Corresponding author.

0304-4149/99/$ – see front matter
PII: S0304-4149(98)00096-9

this paper, as a first step, is to consider a particular class of SPDEs connected with the stochastic Volterra equation in the plane and establish a large deviations principle in the limit of small perturbations of the driving noise in an appropriate anisotropic Besov–Orlicz space.

Rovira and Sanz-Solé (1997) (referred to as [RS] in the sequel) consider the following stochastic Volterra-type equation on the plane.

$$X_z = x_z + \int_{R_z} [\sigma(z;\eta,X_\eta)\, W(\mathrm{d}\eta) + b(z;\eta,X_\eta)\,\mathrm{d}\eta], \tag{1}$$

where $z \in I^2 := [0,1]^2$ and $W = \{W_z;\ z \in I^2\}$ is a Brownian sheet.

An important special case of (1) that appears in Norris (1995) and Rovira and Sanz-Solé (1996) is the following SPDE:

$$\mathscr{L}X_{s,t} = a_3((s,t),X_{s,t})\dot{W}_{s,t} + a_4((s,t),X_{s,t}), \tag{2}$$

where $\dot{W}_{s,t}$ is the white noise on I^2 and

$$\mathscr{L} = \frac{\partial^2}{\partial s \partial t} - a_1(s,t)\frac{\partial}{\partial s} - a_2(s,t)\frac{\partial}{\partial t}.$$

If $(\gamma_{s,t}(\eta),\ \eta \in [0,s] \times [0,t])$ denotes the Green function associated to $\mathscr{L}$, Rovira and Sanz-Solé (1996) show that the solution of Eq. (2) admits the following integral form:

$$X_{s,t} = X_0 + \int_{R_{s,t}} \gamma_{s,t}(\eta)[a_3(\eta,X_\eta)W(\mathrm{d}\eta) + a_4(\eta,X_\eta)\mathrm{d}\eta].$$

A large deviation principle (LDP) for the solutions of Eq. (1), in the limit of small perturbations of the noise, has been established in [RS] in the uniform norm, generalising their previous result for Eq. (2), (see Rovira and Sanz-Solé (1996)) and the one by Doss and Dozzi (1987) for the Brownian sheet. In the Hölder space $\mathscr{C}_\alpha$ with exponent $\alpha < 1/2$, Eddahbi (1997) proved an LDP result for a class of nonlinear SPDEs similar to Eq. (2).

As mentioned above, in this paper, we go one step further and study small perturbations of Eq. (1) in the stronger topology induced by the separable Besov–Orlicz space $\mathscr{B}^0_{\tau,\omega}$ modulated by the Young function $\tau(t) = \exp(t^2) - 1$ and the modulus of continuity $\omega(t) = (t(1+\log(1/t)))^{1/2}$ (see the definition below).

In Section 2, we collect some facts about the Besov–Orlicz space we are concerned with and state the main results. A result of the regularity of the solutions to Eq. (1) in the Besov–Orlicz space is given in Section 3. In Section 4, we prove a large deviation principle in the Besov–Orlicz norm, for the solution of Eq. (1), in the limit of small perturbations of the noise.

2. Preliminaries and main results

2.1. Main results

On the filtered probability space $(\Omega,\ \mathscr{F},\ (\mathscr{F}_z)_{z \in I^2},\ P)$, let $W = \{W_z;\ z = (s,t) \in I^2\}$ be a one-dimensional Brownian sheet, where $(\mathscr{F}_z)_{z \in I^2}$ is the natural filtration associated with W. Let R_z denote the rectangle $[0,z]$, R_{z_1,z_2} the rectangle $[z_1,z_2]$ and $\Gamma(\cdot)$ the Lebesgue measure on $\mathbb{R}^2$.

This paper is devoted to the study of the equation:

$$X_z = x_z + \int_{R_z} [\sigma(z;\eta,X_\eta)\, W(\mathrm{d}\eta) + b(z;\eta,X_\eta)\,\mathrm{d}\eta],$$

under the following conditions:

(H1) b, $\sigma : I^2 \times I^2 \times \mathbb{R} \to \mathbb{R}$ are bounded. σ is uniformly Lipschitz in the second variable.

(H2) σ, b, $\partial\sigma/\partial s$, $\partial\sigma/\partial t$ and $\partial^2\sigma/\partial s\partial t$ are continuous and satisfy the following conditions: There exist positive increasing functions φ and $\tilde{\varphi}$ on $[0,1]$ (typically $\varphi(r) = r^\alpha$, $\alpha \geqslant \frac{1}{2}$) such that

$$\varphi(r) = \mathrm{o}(\omega(r)), \quad r \to 0$$

and $\tilde{\varphi}$ is continuous, such that

(i) For every $z_1, z_2 \in I^2$, $x, y \in \mathbb{R}$ and $\eta_1, \eta_2 \in I^2$,

$$\sup_{\eta \in I^2} |f(R_{z_1,z_2},\eta,x) - f(R_{z_1,z_2},\eta,y)| \leqslant L\varphi(\Gamma(R_{z_1,z_2}))|x-y|,$$

$$\sup_{x \in \mathbb{R}} |f(R_{z_1,z_2},\eta_1,x) - f(R_{z_1,z_2},\eta_2,x)| \leqslant L\varphi(\Gamma(R_{z_1,z_2}))\tilde{\varphi}(|\eta_2-\eta_1|).$$

(ii) For every $s_1, s_2 \in I$, $x, y \in \mathbb{R}$ and $\eta_1, \eta_2 \in I^2$,

$$\begin{aligned}\sup_{t \in I,\ \eta \in I^2} &|f((s_2,t),\eta,x) - f((s_2,t),\eta,y) - f((s_1,t),\eta,x) + f((s_1,t),\eta,y)| \\ &\leqslant L\varphi(|s_2-s_1|)|x-y|,\end{aligned}$$

$$\begin{aligned}\sup_{t \in I,\ x \in \mathbb{R}} &|f((s_2,t),\eta_1,x) - f((s_2,t),\eta_2,x) - f((s_1,t),\eta_1,x) + f((s_1,t),\eta_2,x)| \\ &\leqslant L\varphi(|s_2-s_1|)\tilde{\varphi}(|\eta_2-\eta_1|).\end{aligned}$$

(iii) For every $t_1, t_2 \in I$, $x, y \in \mathbb{R}$ and $\eta_1, \eta_2 \in I^2$,

$$\begin{aligned}\sup_{s \in I,\ \eta \in I^2} &|f((s,t_2),\eta,x) - f((s,t_2),\eta,y) - f((s,t_1),\eta,x) + f((s,t_1),\eta,y)| \\ &\leqslant L\varphi(|t_2-t_1|)|x-y|.\end{aligned}$$

$$\begin{aligned}\sup_{s \in I,\ x \in \mathbb{R}} &|f((s,t_2),\eta_1,x) - f((s,t_2),\eta_2,x) - f((s,t_1),\eta_1,x) + f((s,t_1),\eta_2,x)| \\ &\leqslant L\varphi(|t_2-t_1|)\tilde{\varphi}(|\eta_2-\eta_1|).\end{aligned}$$

Here, f denotes any of the above continuous functions, $f(R_{z_1,z_2}) = f(s_2,t_2) - f(s_1,t_2) - f(s_2,t_1) + f(s_1,t_1)$, for $z_1 = (s_1,t_1)$ and $z_2 = (s_2,t_2)$ and L denotes a constant that may differ from line to line.

(H3) The function $x : I^2 \to \mathbb{R}$ is continuous.

We note that Condition (H2) implies that b and σ are globally Lipschitz in the last argument. Under these conditions, it can be shown (see [RS]) that Eq. (1) admits a unique solution $X = \{X_z,\ z \in I^2\}$ that is $(\mathscr{F}_z)_{z \in I^2}$-adapted with a.s. continuous sample paths.

Let $\mathscr{B}^0_{\tau,\omega}$ denote the separable anisotropic Besov–Orlicz space on I^2, modulated by the Young function $\tau(\cdot)$ and the function $\omega(\cdot)$, with norm $||\cdot||_{\tau,\omega}$ (see the definition below). The following theorem, to be proved in Section 3, is a regularity result for the solution of Eq. (1).

Theorem 1. *Assume that Conditions* (H1) *and* (H2) *are satisfied. If the function x is in $\mathscr{B}^0_{\tau,\omega}$, then the solution X of* Eq. (1) *is almost surely in $\mathscr{B}^0_{\tau,\omega}$.*

The next theorem to be proved in Section 4, is a large deviation principle on $\mathscr{B}^0_{\tau,\omega}$ for the family $\{X^\varepsilon,\ \varepsilon > 0\}$ defined by

$$X^\varepsilon_z = x_z + \sqrt{\varepsilon}\int_{R_z} \sigma(z;\eta,X^\varepsilon_\eta)\, W(\mathrm{d}\eta) + \int_{R_z} b(z;\eta,X^\varepsilon_\eta)\,\mathrm{d}\eta, \tag{3}$$

provided that the function x is in $\mathscr{B}^0_{\tau,\omega}$. Let $\mathscr{H}$ be the Cameron–Martin space associated with the Brownian sheet W.

$$\mathscr{H} := \left\{ h \in \mathscr{C}(I^2,\mathbb{R});\ \text{there exists } \dot{h} \in L^2(I^2,\mathbb{R}) \text{ s.t. } h_z = \int_{R_z} \dot{h}_\eta\ \mathrm{d}\eta,\ z\in I^2 \right\}.$$

Denote

$$||h||^2_{\mathscr{H}} = \int_{I^2} |\dot{h}_\eta|^2\,\mathrm{d}\eta.$$

The rate function associated with the LDP for the Brownian sheet in the uniform norm is then

$$\mu(h) = \begin{cases} \frac{1}{2}||h||^2_{\mathscr{H}}, & \text{if } h\in\mathscr{H}, \\ \mu(h) = +\infty, & \text{otherwise.}\end{cases}$$

Recall that, for every $a \geqslant 0$, the level set of μ

$$L(a) = \{h\in\mathscr{H};\ \mu(h)\leqslant a\}$$

is a compact subset of $\mathscr{H}$. For every $h\in\mathscr{H}$ and every $\gamma\in\mathscr{C}(I^2,\mathbb{R})$, let $S^\gamma(h)$ be the solution of the deterministic differential equation, called the skeleton of Eq. (3):

$$S^\gamma(h)_z = \gamma_z + \int_{R_z} \sigma(z;\eta,S^\gamma(h)_\eta)\dot{h}_\eta\,\mathrm{d}\eta + \int_{R_z} b(z;\eta,S^\gamma(h)_\eta)\,\mathrm{d}\eta \tag{4}$$

and set

$$\lambda(f) = \begin{cases} \inf\{\mu(h);\ S^\gamma(h) = f\}, & \text{if } (S^\gamma)^{-1}(f)\neq\emptyset, \\ +\infty, & \text{otherwise.}\end{cases}$$

Our objective is to prove the following result:

Theorem 2. *Assume* (H1) *and* (H2) *and let the function x be in $\mathscr{B}^0_{\tau,\omega}$. Then the family $\{X^\varepsilon,\ \varepsilon>0\}$ of solutions of* Eq. (3) *satisfies a large deviation principle on $\mathscr{B}^0_{\tau,\omega}$ with rate function*

$$\lambda(f) = \inf\{\tfrac{1}{2}||h||^2_{\mathscr{H}},\ \ S^x(h) = f,\ h\in\mathscr{H}\},$$

with $S^x(h)$ *given by* Eq. (4). *This means that the probabilities* $\mathscr{P}^\varepsilon(A) := P(X^\varepsilon \in A)$, *for measurable sets* A *in* $\mathscr{B}^0_{\tau,\omega}$, *satisfy*

(i) *for each open subset* G *of* $\mathscr{B}^0_{\tau,\omega}$ $\liminf_{\varepsilon\to 0} \varepsilon \log \mathscr{P}^\varepsilon(G) \geqslant -\inf_{f\in G} \lambda(f)$,

(ii) *for each closed subset* F *of* $\mathscr{B}^0_{\tau,\omega}$ $\limsup_{\varepsilon\to 0} \varepsilon \log \mathscr{P}^\varepsilon(F) \leqslant -\inf_{f\in F} \lambda(f)$.

(iii) *The level sets* $\{f \in \mathscr{B}^0_{\tau,\omega};\, \lambda(f) \leqslant N\}$, $N > 0$, *are compact.*

To prove the LDP, we apply the transfer principle deviced by Azencott (1980) (see also [RS]).

Let us remark that these results extend easily to the more general situations where the coefficients may depend on ε provided some suitable convergence assumptions.

2.2. Besov–Orlicz space

In this subsection, we give a short review of the Besov–Orlicz space we consider in this paper. For more details on these spaces, see Rao and Ren (1991), Peetre (1976), Jonsson and Wallin (1984), Ciesielski et al. (1993) and Ropela (1976) for the one-parameter case and Ciesielski and Domsta (1972), Ciesielski and Kamont (1995) and Kamont (1994, 1996) for the multiparameter case. This review follows closely Ciesielski and Kamont (1995) and Kamont (1994).

The Orlicz space $L(\tau, I^2)$ on I^2 associated to τ is the space of measurable functions $f : I^2 \to \mathbb{R}$ for which

$$||f||^*_\tau := \inf_{\beta>0} \frac{1}{\beta}\left[1 + \int_{I^2} \tau(\beta|f(t)|)\,\mathrm{d}t\right] < \infty.$$

For the purpose of the paper it is more convenient to use an equivalent norm to $||\cdot||^*_\tau$ (see Fernique (1971) or Ciesielski (1993) for a proof):

$$||f||_\tau := \sup_{p\geqslant 1} \frac{||f||_p}{\sqrt{p}}, \tag{5}$$

where, $||f||_p$ is the usual $L^p(I^2)$-norm. For $r \in \mathbb{R}$ and $e_i = (\delta_{1,i}, \delta_{2,i}) \in \mathbb{R}^2$, $i = 1, 2$, unit vectors, let

$$\Delta_{r,i} f(z) = \begin{cases} f(z + re_i) - f(z), & \text{if } z \text{ and } z + re_i \in I^2, \\ 0, & \text{otherwise,} \end{cases}$$

denote the progressive difference of a measurable function f on I^2 in the e_ith direction. Set

$$\Delta_{(r_1,r_2)} f = \Delta_{r_1,1} \circ \Delta_{r_2,2} f, \quad (r_1, r_2) \in \mathbb{R}^2$$

and define the moduli of smoothness of f in $L^p(I^2)$ by

$$\omega_{p,i}(f, t) := \sup_{|r|\leqslant t} ||\Delta_{r,i} f||_p, \quad i = 1, 2, \quad t \in (0, 1]$$

and

$$\omega_p(f, (t_1, t_2)) := \sup_{|r_1|\leqslant t_1, |r_2|\leqslant t_2} ||\Delta_{(r_1,r_2)} f||_p, \quad t_1, t_2 \in (0, 1].$$

Using Eq. (5), the moduli of smoothness of f in $L(\tau, I^2)$ are defined by the usual formulas:

$$\omega_{\tau,i}(f,t) := \sup_{p\geqslant 1} \frac{\omega_{p,i}(f,t)}{\sqrt{p}} \tag{6}$$

and

$$\omega_{\tau}(f,(t_1,t_2)) := \sup_{p\geqslant 1} \frac{\omega_{p}(f,(t_1,t_2))}{\sqrt{p}}. \tag{7}$$

Consider the function ω, sometimes called $(0,1)$-*modulus*, defined on I by

$$\omega(t) = \sqrt{t\left(1+\log\frac{1}{t}\right)}.$$

An anisotropic Besov–Orlicz space $\mathscr{B}_{\tau,\omega}$ is a Besov space modulated with the Orlicz norm described in terms of $\omega(\cdot)$ and the moduli of smoothness defined in Eqs. (6) and (7):

$$\mathscr{B}_{\tau,\omega} := \{f \in L(\tau, I^2);\ ||f||_{\tau,\omega} < +\infty\},$$

where

$$||f||_{\tau,\omega} := \sup_{0<t\leqslant 1} \frac{\omega_{\tau,1}(f,t)}{\omega(t)} + \sup_{0<t\leqslant 1} \frac{\omega_{\tau,2}(f,t)}{\omega(t)} + \sup_{0<t_1,t_2\leqslant 1} \frac{\omega_{\tau}(f,(t_1,t_2))}{\omega(t_1)\omega(t_2)} + ||f||_{\tau}.$$

$\mathscr{B}_{\tau,\omega}$ endowed with the norm $||\cdot||_{\tau,\omega}$ is a nonseparable Banach space. We are going to consider a separable Banach subspace of $\mathscr{B}_{\tau,\omega}$ defined as follows:

$$\begin{aligned}\mathscr{B}^0_{\tau,\omega} := \Big\{ & f \in \mathscr{B}_{\tau,\omega};\ ||f||_p = \mathrm{o}(\sqrt{p}) \text{ as } p \to +\infty,\\ & \omega_p(f,(t_1,t_2)) = \mathrm{o}(\sqrt{p}\ \omega(t_1)\omega(t_2)) \text{ as } \frac{1}{p}\wedge t_1 \wedge t_2 \to 0,\\ & \omega_{p,i}(f,t) = \mathrm{o}(\sqrt{p}\ \omega(t)) \text{ as } \frac{1}{p}\wedge t \to 0 \text{ for } i=1,2 \Big\}.\end{aligned}$$

The fact that these spaces are isomorphic to some sequence spaces makes them relatively easy to manipulate. The isomorphism is given by the coefficients of a function in the tensor product Schauder system (see Theorem A in Kamont (1994)). Indeed, let $\{\varphi_n,\ n\geqslant 0\}$ be the family of Schauder functions on I defined by

$$\begin{aligned}&\varphi_0(s) = 1,\ \ \varphi_1(s) = s,\\ &\varphi_n(s) = \sqrt{2^{-j}}\varphi(2^{j+1}s - k) \text{ for } n = 2^j + k,\ j \in \mathbb{N} \text{ and } k = 1,\ldots,2^j.\end{aligned}$$

With $\varphi(u) = \max(1-|u|, 0)$. We know that for each continuous function f on I, we have

$$f(s) = \sum_{n=0}^{+\infty} C_n(f)\varphi_n(s),$$

where the coefficients are given by

$$\begin{aligned}&C_0(f) = f(0), \quad C_1(f) = f(1) - f(0),\\ &C_n(f) = 2\sqrt{2^j}\left[f\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2}\left(f\left(\frac{2k}{2^{j+1}}\right) + f\left(\frac{2k-2}{2^{j+1}}\right)\right)\right] \text{ for } n = 2^j + k.\end{aligned}$$

Now, if f is continuous in I^2 we have the decomposition

$$f=\sum_{m=0}^{+\infty}\sum_{n\vee n'=m}^{+\infty} C_{n,n'}(f)\varphi_n\otimes\varphi_{n'}$$

and the coefficients are given by

$$C_{n,n'}(f)=C_n^1(f)\circ C_{n'}^2(f),$$

with

$$C_n^1(f)(x_2)=C_n(f(\cdot,x_2)) \quad \text{and} \quad C_n^2(f)(x_1)=C_n(f(x_1,\cdot)).$$

The main tools to prove our results rely on the following characterisation of the space $\mathscr{B}^0_{\tau,\omega}$ (see Ciesielski and Kamont, 1995):

Theorem 3. *$f\in\mathscr{B}^0_{\tau,\omega}$ if and only if conditions* (A1), (A2) *and* (A3) *below are satisfied.*

$$\text{(A1)}\qquad \lim_{j'\vee p\to+\infty}\frac{2^{-j'/p}}{\sqrt{p(1+j')}}\left(\sum_{n'=2^{j'}+1}^{2^{j'+1}}|C_{l,n'}(f)|^p\right)^{1/p}=0,\quad l=0,1.$$

$$\text{(A2)}\qquad \lim_{j\vee p\to+\infty}\frac{2^{-j/p}}{\sqrt{p(1+j)}}\left(\sum_{n=2^{j}+1}^{2^{j+1}}|C_{n,l'}(f)|^p\right)^{1/p}=0,\quad l'=0,1.$$

$$\text{(A3)}\qquad \lim_{j\vee j'\vee p\to+\infty}\frac{2^{-(j+j')/p}}{\sqrt{p(1+j)(1+j')}}\left(\sum_{n=2^{j}+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|C_{n,n'}(f)|^p\right)^{1/p}=0.$$

Two other norms $||\cdot||_*$ and $||\cdot||^{**}$ (inducing nonseparable Banach spaces) that play a crucial role in the proof of the LDP results are defined as follows. For $f: I^2\to\mathbb{R}$, vanishing on the axes, set

$$||f||_*=\max(|f(1,1)|,||f||_{*1},||f||_{*2},||f||_{*3})$$

with

$$||f||_{*1}=\sup_{j\geqslant 0}\ \sup_{2^j+1\leqslant n\leqslant 2^{j+1}}\frac{|C_n(f(.,1))|}{\sqrt{1+j}},$$

$$||f||_{*2}=\sup_{j\geqslant 0}\ \sup_{2^j+1\leqslant n\leqslant 2^{j+1}}\frac{|C_n(f(1,.))|}{\sqrt{1+j}},$$

$$||f||_{*3}=\sup_{j,j'\geqslant 0}\ \sup_{(n,n')\in\mathscr{K}_{j,j'}}\frac{|C_{n,n'}(f)|}{\sqrt{(1+j)(1+j')}},$$

where, $\mathscr{K}_{j,j'}=\{(n,n');\ 2^j+1\leqslant n\leqslant 2^{j+1},\ 2^{j'}+1\leqslant n'\leqslant 2^{j'+1}\}$; and

$$||f||^{**}=||f||_1^{**}+||f||_2^{**}+||f||_3^{**},$$

with

$$||f||_1^{**} = \sup_{0 \leqslant s_1 < s_2 \leqslant 1} \frac{|f(s_1,1) - f(s_2,1)|}{\omega(|s_1 - s_2|)},$$

$$||f||_2^{**} = \sup_{0 \leqslant t_1 < t_2 \leqslant 1} \frac{|f(1,t_2) - f(1,t_1)|}{\omega(|t_1 - t_2|)},$$

and

$$||f||_3^{**} = \sup_{\substack{0 \leqslant t_1 < t_2 \leqslant 1 \\ 0 \leqslant s_1 < s_2 \leqslant 1}} \frac{|f(R_{z_1,z_2})|}{\omega(|s_1 - s_2|)\omega(|t_1 - t_2|)}.$$

In particular, for rectangles R_z, $z = (s,t) \in I^2$, and functions f null on the axes, we have $f(s,t) = f(R_z)$ and therefore

$$||f|| \leqslant ||f||^{**}, \tag{8}$$

where, $||\cdot||$ denotes the usual uniform norm for the corresponding space. The importance of these norms relies on the fact that

$$||f|| \leqslant C_0 ||f||_{\tau,\omega} \leqslant C_1 ||f||^{**} \leqslant C_2 ||f||_*,$$

for some positive constants C_0, C_1 and C_2, where the last inequality holds only for functions f being null on the axes.

2.3. Exponential inequalities

In this subsection, we list the exponential inequalities for continuous strong martingales and stochastic integrals (that are not martingales) in the plane we will frequently use in the sequel. The first inequality appears in Proposition 7 in Dozzi (1989, p. 114) (cf. also Theorem 2.4 in Rovira and Sanz-Solé (1997) and Mishura (1987)). The proofs of the remaining propositions can be found in Appendix A.

Proposition 4. *Let $M = \{M_z; z \in I^2\}$ be a continuous strong $\mathscr{F}_z$-martingale null on the axes. Assume that there exists a function a: $\mathbb{R}^2 \to (0, +\infty)$ such that $P(\langle M \rangle_z > a(z)) = 0$ for all $z \in I^2$, where $\{\langle M \rangle_z,\ z \in I^2\}$ is the quadratic variation of M. Then, for any $u > 0$*

$$P\left(\sup_{\eta \in R_z} |M_\eta| \geqslant u\right) \leqslant 4 \exp\left(-\frac{u^2}{18a(z)}\right).$$

Proposition 5. *For every $u > 12\sqrt{\log 2}$ and every $\mathbb{R}$-valued continuous and bounded process $H = \{H(\eta),\ \eta \in I^2\}$*

$$P\left[\left|\left|\int_{R_\cdot} H(\eta) W(\mathrm{d}\eta)\right|\right|_* \geqslant u,\ ||H|| \leqslant 1\right] \leqslant 100 \exp\left(-\frac{u^2}{144}\right),$$

and therefore the inequality holds for the norms $||\cdot||_{\tau,\omega}$ and $||\cdot||^{**}$ instead of $||\cdot||_*$.

The next two propositions are a $\mathscr{B}^0_{\tau,\omega}$-version of Theorems 2.3 and 2.4 in [RS].

Proposition 6. *Let $H : I^2 \times I^2 \times \Omega \to \mathbb{R}$ be a $\mathscr{B}(I^2) \otimes \mathscr{B}(I^2) \otimes \mathscr{F}$-measurable process satisfying*:

(a) *For each $z, H(z,\eta)$ is $\mathscr{F}_\eta$-adapted for which $E[\int_{I^2} H^2(z,\eta)\,\mathrm{d}\eta] < \infty$.*

(b) *The map $z \to H(z,\cdot)$ satisfies a.s. Assumption* (H2).

Then, there exist positive constants C_1 and C_2, independent of L, such that

(i)

$$P\left[\left|\left|\int_{R.} H(\cdot,\eta)W(\mathrm{d}\eta)\right|\right|_* > \rho;\ ||H|| \leqslant \alpha\right] \leqslant \exp -\left(\frac{\rho^2 C_2}{12L^2 + 4\alpha^2}\right) \tag{9}$$

for all ρ for which $\rho \geqslant 2C_1\sqrt{3L^2+\alpha^2}$.

(ii)

$$P\left[\left|\left|\int_{R.}\int_{R_{\eta,\cdot}} H(\xi,\eta)\,\mathrm{d}\xi W(\mathrm{d}\eta)\right|\right|_* > \rho,\ ||H|| \leqslant \alpha\right] \leqslant \exp -\left(\frac{\rho^2 C_2}{12L^2 + 4\alpha^2}\right). \tag{10}$$

Consequently, inequalities (9) *and* (10) *hold for the norms $||\cdot||_{\tau,\omega}$ and $||\cdot||^{**}$ instead of $||\cdot||_*$.*

Denote for a $\mathscr{B}(I^2) \otimes \mathscr{B}(I) \otimes \mathscr{F}$-measurable process H

$$(H \cdot Ws)(s,t) := \int_0^s \int_{R_{s,t}} H(\eta,r)W(\mathrm{d}\eta)\mathrm{d}r$$

and for a $\mathscr{B}(I) \otimes \mathscr{B}(I^2) \otimes \mathscr{F}$-measurable process $\tilde{H}$

$$(\tilde{H} \cdot tW)(s,t) := \int_0^t \int_{R_{s,t}} \tilde{H}(r,\eta)W(\mathrm{d}\eta)\mathrm{d}r$$

Proposition 7. (i) *Let $H : I^2 \times I \times \Omega \to \mathbb{R}$ be a $\mathscr{B}(I^2) \otimes \mathscr{B}(I) \otimes \mathscr{F}$-measurable process satisfying*

$$E\left[\int_0^1 \int_{I^2} H^2(\eta,r)\,\mathrm{d}\eta\,\mathrm{d}r\right] < \infty.$$

If for each $\eta=(u,v)$ and $z=(s,t), H((u,v),s)$ is $\mathscr{F}_{(s,v)}$-measurable and $H((u,v),s)$ vanishes for $0 \leqslant s < u$, then there exists a positive constant C such that

$$P\left[||(H \cdot Ws)(\cdot,\cdot)||^{**} > \rho,\ ||H|| \leqslant \alpha\right] \leqslant C\exp -\left(\frac{\rho^2}{C(\alpha^2+L^2)}\right). \tag{11}$$

(ii) *Let $\tilde{H} : I \times I^2 \times \Omega \to \mathbb{R}$ be a $\mathscr{B}(I) \otimes \mathscr{B}(I^2) \otimes \mathscr{F}$-measurable process satisfying*

$$E\left[\int_0^1 \int_{I^2} \tilde{H}^2(r,\eta)\,\mathrm{d}\eta\,\mathrm{d}r\right] < \infty.$$

If for each $\eta=(u,v)$ and $z=(s,t), \tilde{H}(t,(u,v))$ is $\mathscr{F}_{(u,t)}$-measurable and $H(t,(u,v))$ vanishes for $0 \leqslant t < v$, then there exists a positive constant C such that

$$P[||(\tilde{H} \cdot tW)(\cdot,\cdot)||^{**} > \rho,\ ||H|| \leqslant \alpha] \leqslant C\exp -\left(\frac{\rho^2}{4C(\alpha^2+3L^2)}\right). \tag{12}$$

Consequently, inequalities (11) *and* (12) *hold for the norm $||\cdot||_{\tau,\omega}$ instead of $||\cdot||^{**}$.*

3. Regularity of the solutions in the Besov–Orlicz space

This section is devoted to the proof of Theorem 1 on the regularity of the process X, solution of Eq. (1), in $\mathscr{B}^0_{\tau,\omega}$. Recall

$$X_z = x_z + \int_{R_z} [\sigma(z;\eta,X_\eta)\, W(\mathrm{d}\eta) + b(z;\eta,X_\eta)\,\mathrm{d}\eta],$$

where σ and b satisfy Conditions (H1) and (H2) above and the function $x : I^2 \to \mathbb{R}$ is in $\mathscr{B}^0_{\tau,\omega}$.

For the sequel, we denote $\tilde{L} = \mathrm{Lip}(\sigma) \vee \mathrm{Lip}(b)$, the largest of the Lipschitz constants of the coefficients and $M = ||\sigma|| \vee ||b||$, where $||\cdot||$ denotes the usual uniform norm for the corresponding spaces.

The proof of Theorem 1 consists in checking Conditions (A1)–(A3) in Theorem 2.1 above. First we need an L^p-estimate for the stochastic integral

$$Y_z := \int_{R_z} \sigma(z;\eta,X_\eta) W(\mathrm{d}\eta).$$

We note that for $\eta := (u,v) \leqslant z := (s,t)$ and $\xi := (r,w)$, our stochastic integral can be written as a *representable semimartingale* (cf. Eq. (1.5) in [RS]):

$$Y_z = J^1_z + J^2_z + J^3_z + J^4_z,$$

where

$$J^1_z = \int_{R_z} \sigma(\eta;\eta,X_\eta) W(\mathrm{d}\eta),$$

$$J^2_z = \int_0^s \int_{R_z} \frac{\partial\sigma}{\partial s}((r,v);\eta,X_\eta) W(\mathrm{d}\eta)\,\mathrm{d}r,$$

$$J^3_z = \int_0^t \int_{R_z} \frac{\partial\sigma}{\partial t}((u,w);\eta,X_\eta) W(\mathrm{d}\eta)\,\mathrm{d}w$$

and

$$J^4_z = \int_{R_z} \int_{R_{\eta,z}} \frac{\partial^2\sigma}{\partial s\partial t}(\xi;\eta,X_\eta)\,\mathrm{d}\xi\, W(\mathrm{d}\eta).$$

Lemma 8. *For every integer $p \geqslant 2$,*

$$E|Y_z|^p \leqslant C^p \; p^{p/2} \; \Gamma(R_z)^{p/2}, \tag{13}$$

where C is a positive constant.

Proof (*Sketch*)**.** The idea of the proof uses arguments applied to the uniform case in [RS], the two-parameters Itô's formula and Burkholder–Davis–Gundy inequality to each of the $(J^i_z)^p$'s above. □

Proof of Theorem 1 (*Sketch*)**.** First we remark from the definition of $\mathscr{B}^0_{\tau,\omega}$ and (H2) that $z \to \int_{R_z} b(z;\eta,X_\eta)\mathrm{d}\eta$ belongs a.s to $\mathscr{B}^0_{\tau,\omega}$. Indeed it is not hard to show that this map belongs a.s. to $\mathscr{B}_{\tau,\omega}$. The fact that it belongs to $\mathscr{B}^0_{\tau,\omega}$ follows from the assumption that $\varphi(r) = o(\omega(r))$ as $r \to 0$. It remains to show that the process $(Y_z,\ z \in I^2)$ satisfies (A1)–(A3) of Theorem 3. We will only prove (A3). (A1) and (A2) can be derived in the same fashion and therefore we omit the details.

To prove (A3) for J_z^i, $i = 1, \dots, 4$ we shall show that for all $0 < \alpha < \frac{1}{2}$,

$$\text{(B1)} \qquad \sup_{j,j' \geqslant 0} \sup_{p \geqslant 1} \frac{2^{-(j+j')/p}}{\sqrt{p}(1+j)^{\alpha}(1+j')^{\alpha}} \left(\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(J_\cdot^i)|^p \right)^{1/p} < +\infty \quad \text{a.s.}$$

and

$$\text{(B2)} \qquad \lim_{p \to +\infty} \sup_{j,j' \geqslant 0} \frac{2^{-(j+j')/p}}{\sqrt{p}\sqrt{(1+j)(1+j')}} \left(\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(J_\cdot^i)|^p \right)^{1/p} = 0 \quad \text{a.s.}$$

To check relation (B1), let $A > 0$. Using Chebyshev's inequality, we get

$$P\left[\frac{2^{-(j+j')/p}}{\sqrt{p}(1+j)^{\alpha}(1+j')^{\alpha}} \left(\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(J_\cdot^i)|^p \right)^{1/p} > A \right]$$

$$\leqslant \frac{2^{-(j+j')}}{A^p(\sqrt{p})^p(1+j)^{\alpha p}(1+j')^{\alpha p}} \left(\sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} E|C_{n,n'}(J_\cdot^i)|^p \right).$$

Let us first estimate the expectation of the pth power of $C_{n,n'}(J_\cdot^4)$. Recall that

$$J_z^4 = \int_{R_z} \int_{R_{\eta,z}} \frac{\partial^2 \sigma}{\partial s \partial t}(\xi; \eta, X_\eta) \mathrm{d}\xi \, W(\mathrm{d}\eta) := \int_{R_z} H(z,\eta) W(\mathrm{d}\eta)$$

with

$$H(z,\eta) := \int_{R_{\eta,z}} \frac{\partial^2 \sigma}{\partial s \partial t}(\xi, \eta, X_\eta) \mathrm{d}\xi.$$

By definition, $|C_{n,n'}(J_\cdot^4)|$ is less than four terms of the form

$$2^{(j+j')/2}(D_1 + D_2 + D_3 + D_4),$$

where

$$D_1 = \left| \int_0^{(k-1)/2^j} \int_0^{(k'-1)/2^{j'}} H(R_{((k-1)/2^j, k/2^j),((k'-1)/2^{j'}, k'/2^{j'})}, \eta) W(\mathrm{d}\eta) \right|,$$

$$D_2 = \left| \int_0^{(k-1)/2^j} \int_{(k'-1)/2^{j'}}^{k'/2^{j'}} \left(H\left(\left(\frac{k}{2^j}, \frac{k'}{2^{j'}} \right), \eta \right) - H\left(\left(\frac{k-1}{2^j}, \frac{k'}{2^{j'}} \right), \eta \right) \right) W(\mathrm{d}\eta) \right|,$$

$$D_3 = \left| \int_{(k-1)/2^j}^{k/2^j} \int_{(k'-1)/2^{j'}}^{k'/2^{j'}} H\left(\left(\frac{k}{2^j}, \frac{k'}{2^{j'}} \right), \eta \right) W(\mathrm{d}\eta) \right|,$$

$$D_4 = \left| \int_{(k-1)/2^j}^{k/2^j} \int_0^{(k'-1)/2^{j'}} \left(H\left(\left(\frac{k}{2^j}, \frac{k'}{2^{j'}} \right), \eta \right) - H\left(\left(\frac{k}{2^j}, \frac{k'-1}{2^{j'}} \right), \eta \right) \right) W(\mathrm{d}\eta) \right|.$$

Hence,

$$
\begin{aligned}
&E|C_{n,n'}(J^4_\cdot)|^p \\
&\leqslant 4^p\, 2^{(j+j')p/2} \sup_{\substack{|t'-s'|\leqslant 2^{-j'-1}\\ |t-s|\leqslant 2^{-j-1}}} E\left|\int_0^s\int_0^{s'} H(R_{(s,t),(s',t')},\eta)W(\mathrm{d}\eta)\right|^p \\
&\quad + 4^p 2^{(j+j')p/2} \sup_{\substack{|t'-s'|\leqslant 2^{-j'-1}\\ |t-s|\leqslant 2^{-j-1}}} E\left|\int_s^t\int_{s'}^{t'} H((t,t'),\eta)W(\mathrm{d}\eta)\right|^p \\
&\quad + 4^p 2^{(j+j')p/2} \sup_{\substack{|t'-s'|\leqslant 2^{-j'-1}\\ |t-s|\leqslant 2^{-j-1}}} E\left|\int_0^s\int_{s'}^{t'} (H((t,t'),\eta)-H((s,t'),\eta))W(\mathrm{d}\eta)\right|^p \\
&\quad + 4^p 2^{(j+j')p/2} \sup_{\substack{|t'-s'|\leqslant 2^{-j'-1}\\ |t-s|\leqslant 2^{-j-1}}} E\left|\int_s^t\int_0^{s'} (H((t,t'),\eta)-H((t,s'),\eta))W(\mathrm{d}\eta)\right|^p.
\end{aligned}
$$

Therefore by Eq. (13), for integers $p\geqslant 2$,

$$E|C_{n,n'}(J^4_\cdot)|^p \leqslant 4(2M)^p p^{p/2}.$$

Using the multiparameter Fubini theorem, Hölder's inequality and Lemma 8, estimates of the expectation of each of the $C_{n,n'}(J^i_\cdot)$'s can be checked in a similar way. Therefore,

$$\max_{i=1,2,3,4} E|C_{n,n'}(J^i_\cdot)|^p \leqslant CM^p p^{p/2}.$$

Hence,

$$
\begin{aligned}
&P\left[\frac{2^{-(j+j')/p}}{\sqrt{p}(1+j)^\alpha(1+j')^\alpha}\left(\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(J^i_\cdot)|^p\right)^{1/p} > A\right] \\
&\leqslant 4\left(\frac{M}{2A}\right)^p \frac{1}{(1+j)^{\alpha p}(1+j')^{\alpha p}}.
\end{aligned}
$$

Choosing $p_0 > 1/\alpha$ and A large enough, the series

$$\sum_{j,j'\geqslant 0}\sum_{p\geqslant p_0}\left(\frac{M}{2A}\right)^p \frac{1}{(1+j)^{\alpha p}(1+j')^{\alpha p}}$$

converges. Relation (B1) is then a consequence of Borel–Cantelli lemma.

To check relation (B2), let

$$\mathscr{K}_{j,j'} = \{(n,n');\ 2^j+1\leqslant n\leqslant 2^{j+1},\ 2^{j'}+1\leqslant n'\leqslant 2^{j'+1}\}$$

and note that

$$2^{-(j+j')/p}\left(\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(J^i_\cdot)|^p\right)^{1/p} \leqslant \sup_{(n,n')\in\mathscr{K}_{j,j'}} |C_{n,n'}(J^i_\cdot)|.$$

Now, using the fact that each of the $C_{n,n'}(J^i_\cdot)$'s is dominated by terms of the same form as $D_1,\dots,D_4$, the exponential inequalities of Propositions 4–7 yield that there exist positive constants C_1 and C_2 such that for all $A>0$ large enough

$$P\left[\frac{1}{\sqrt{(1+j)(1+j')}}\sup_{(n,n')\in\mathscr{K}_{j,j'}}|C_{n,n'}(J^i_\cdot)|>A\right]$$

$$\leqslant C_1\exp\left(-\left(\frac{A^2}{C_2M^2}\right)(1+j)(1+j')\right).$$

Borel–Cantelli lemma yields then

$$\sup_{j,j'}\frac{1}{\sqrt{(1+j)(1+j')}}\sup_{(n,n')\in\mathscr{K}_{j,j'}}|C_{n,n'}(J^i_\cdot)|<+\infty \text{ a.s.}$$

But

$$\sup_{j,j'}\frac{2^{-(j+j')/p}}{\sqrt{p}\sqrt{(1+j)(1+j')}}\left(\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|C_{n,n'}(J^i_\cdot)|^p\right)^{1/p}$$

$$\leqslant\frac{1}{\sqrt{p}}\sup_{j,j'}\frac{1}{\sqrt{(1+j)(1+j')}}\sup_{(n,n')\in\mathscr{K}_{j,j'}}|C_{n,n'}(J^i_\cdot)|,$$

which completes the proof of relation (A3). □

4. Large deviations in the Besov–Orlicz norm

In this section, we prove Theorem 2. Following the transfer principle deviced by Azencott (see also Rovira and Sanz-Solé (1996, 1997)), the proof of the theorem is a direct consequence of the next two propositions:

Proposition 9. *For every $a\geqslant 0$ and every $x\in\mathscr{B}^0_{\tau,\omega}$, the map*

$$S:\mathscr{H}\to\mathscr{B}^0_{\tau,\omega},$$
$$h\to S^x(h)$$

is continuous in $L(a)$ endowed with the Besov–Orlicz norm $||\cdot||_{\tau,\omega}$. In particular, the level sets $\{f\in\mathscr{B}^0_{\tau,\omega};\ \lambda(f)\leqslant N\}$, $N>0$, are compact.

Proposition 10. *For every $R>0$, $a>0$ and $\rho>0$ large enough, there exist $\alpha>0$ and $\varepsilon_0>0$ such that for every $h\in L(a)$ and every $\varepsilon\in(0,\varepsilon_0]$ we have*

$$P(||X^\varepsilon-S^x(h)||_{\tau,\omega}\geqslant\rho,\ ||\sqrt{\varepsilon}W-h||\leqslant\alpha)\leqslant\exp\left(-\frac{R}{\varepsilon}\right).$$

Proof of Proposition 9. Let $h,k\in\mathscr{H}$ such that $||h||_{\mathscr{H}}\vee||k||_{\mathscr{H}}\leqslant a$. Lemma 3.2 in [RS] tells us that the map $S:h\to S^x(h)_\cdot$ is continuous in the set $L(a)$ with respect to the uniform norm. On the other hand

$$\begin{aligned}||S^x(h)_\cdot - S^x(k)_\cdot||_{\tau,\omega} &\leqslant \left|\left|\int_{R_\cdot}(b(\cdot;\eta,S^x(h)_\eta)-b(\cdot;\eta,S^x(k)_\eta))\mathrm{d}\eta\right|\right|_{\tau,\omega}\\ &\quad+\left|\left|\int_{R_\cdot}(\sigma(\cdot;\eta,S^x(h)_\eta)-\sigma(\cdot;\eta,S^x(k)_\eta))\dot{k}_\eta\,\mathrm{d}\eta\right|\right|_{\tau,\omega},\\ &\quad+\left|\left|\int_{R_\cdot}\sigma(\cdot;\eta,S^x(h)_\eta)(\dot{h}_\eta-\dot{k}_\eta)\,\mathrm{d}\eta\right|\right|_{\tau,\omega}\\ &= I_1+I_2+I_3.\end{aligned}$$

Since I_1 is a particular case of I_2 if we replace, in I_2, σ by b and $\dot{k}_\eta$ by 1, it suffices to prove that I_2 and I_3 tend to zero as $||h-k||_{\tau,\omega}$ tends to zero.

For ease of notation we set $\phi(z,\eta,x,y)=b(z;\eta,x)-b(z;\eta,y)$, $\psi(z,\eta,x,y)=\sigma(z;\eta,x)-\sigma(z;\eta,y)$ and

$$\Psi(z)=\int_{R_z}\psi(z,\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta.$$

Hence,

$$I_2=||\Psi||_\tau+\sup_{0<t\leqslant 1}\frac{\omega_{\tau,1}(\Psi,t)}{\omega(t)}+\sup_{0<t\leqslant 1}\frac{\omega_{\tau,2}(\Psi,t)}{\omega(t)}+\sup_{0<t_1,t_2\leqslant 1}\frac{\omega_\tau(\Psi,(t_1,t_2))}{\omega(t_1)\omega(t_2)}.$$

Using Eq. (5) and the Lipschitz property of σ we get

$$||\Psi||_\tau\leqslant L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}. \tag{14}$$

To estimate the remaining terms of I_2, recall that

$$\omega_{p,i}(\Psi,t)=\sup_{|r|\leqslant t}||\Delta_{r,i}\Psi||_p,\quad i=1,2,\ \ 0<t\leqslant 1,$$

$$\omega_p(\Psi,(t_1,t_2))=\sup_{\substack{|r_2|\leqslant t_2\\|r_1|\leqslant t_1}}||\Delta_{(r_1,r_2)}\Psi||_p,\quad 0<t_1,t_2\leqslant 1,$$

$$\omega_{\tau,i}(\Psi,t)=\sup_{p\geqslant 1}\frac{\omega_{p,i}(\Psi,t)}{\sqrt{p}},\quad 0<t\leqslant 1,\ \ i=1,2$$

and

$$\omega_\tau(\Psi,(t_1,t_2))=\sup_{p\geqslant 1}\frac{\omega_p(\Psi,(t_1,t_2))}{\sqrt{p}},\quad 0<t_1,t_2\leqslant 1.$$

We have

$$||\Delta_{r,i}\Psi||_p=\left(\int_{I^2_{r,i}}|\Psi(z+re_i)-\Psi(z)|^p\,\mathrm{d}z\right)^{1/p},$$

where $I^2_{r,i}=\{z\in I^2:\ z+re_i\in I^2\}$. For $i=1$ and $z=(s,t)$,

$$\begin{aligned}|\Psi(z+re_1)-\Psi(z)|&\leqslant\left|\int_s^{s+r}\int_0^t\psi((s+r,t),\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta\right|\\ &\quad+\left|\int_0^s\int_0^t(\psi((s+r,t),\eta,S^x(h)_\eta,S^x(k)_\eta)-\psi((s,t),\eta,S^x(h)_\eta,S^x(k)_\eta))\dot{k}_\eta\,\mathrm{d}\eta\right|.\end{aligned}$$

Assumption (H2) implies that

$$|\Psi(z+re_1)-\Psi(z)|\leqslant L||S^x(h)-S^x(k)||\left(\int_s^{s+r}\int_0^t |\dot{k}_\eta|\mathrm{d}\eta+\varphi(r)\int_0^s\int_0^t |\dot{k}_\eta|\mathrm{d}\eta\right).$$

Using Cauchy–Schwarz inequality we get

$$|\Psi(z+re_1)-\Psi(z)|\leqslant L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}(\sqrt{rt}+\varphi(r)\sqrt{st}).$$

Similarly,

$$|\Psi(z+re_2)-\Psi(z)|\leqslant L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}(\sqrt{rs}+\varphi(r)\sqrt{st}).$$

Hence,

$$\begin{aligned}||\Delta_{r,1}\Psi||_p+||\Delta_{r,2}\Psi||_p &\leqslant 2L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}\left(\int_{I^2_{r,1}}(\sqrt{rt}+\varphi(r)\sqrt{st})^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}\\ &\leqslant 4\sqrt{r}L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}\left(\int_{I^2_{r,1}} t^{p/2}\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}\\ &\quad +4\varphi(r)L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}\left(\int_{I^2_{r,2}}(st)^{p/2}\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}\\ &\leqslant 4(\sqrt{r}+\varphi(r))L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}.\end{aligned}$$

Therefore,

$$\omega_{p,i}(\Psi,t)=\sup_{|r|\leqslant t}||\Delta_{r,i}\Psi||_p\leqslant 4L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}(\sqrt{t}+\varphi(t))$$

and then

$$\omega_{\tau,i}(\Psi,t)=\sup_{p\geqslant 1}\frac{\omega_{p,i}(\Psi,t)}{\sqrt{p}}\leqslant 4L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}(\sqrt{t}+\varphi(t)),$$

which implies that

$$\sup_{0<t\leqslant 1}\frac{\omega_{\tau,i}(\Psi,t)}{\omega(t)}\leqslant 4L||S^x(h)-S^x(k)||\,||k||_{\mathscr{H}}\,D_1$$

with

$$D_1:=\sup_{0<t\leqslant 1}\frac{\sqrt{t}+\varphi(t)}{\omega(t)},$$

which is finite by assumption on φ. Consequently,

$$\sup_{0<t\leqslant 1}\frac{\omega_{\tau,1}(\Psi,t)}{\omega(t)}+\sup_{0<t\leqslant 1}\frac{\omega_{\tau,2}(\Psi,t)}{\omega(t)}\leqslant 8L||S^x(h)-S^x(k)||\;||k||_{\mathscr{H}}\,D_1. \tag{15}$$

Now, for $r=(r_1,r_2)\in R^2_+$ we have

$$\begin{aligned}|\Delta_{(r_1,r_2)}\Psi(z)| \leqslant& \left|\int_{R_z}\psi(R_{z,z+r},\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta\right|\\ &+\left|\int_{R_{z,z+r}}\psi((z+r),\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^s\int_t^{t+r_2}(\psi((z+r),\eta,S^x(h)_\eta,S^x(k)_\eta)\right.\\
&\left.-\psi((s,t+r_2),\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta\right|\\
&+\left|\int_0^{s+r_1}\int_0^t(\psi((z+r),\eta,S^x(h)_\eta,S^x(k)_\eta)\right.\\
&\left.-\ \psi((s+r_1,t),\eta,S^x(h)_\eta,S^x(k)_\eta)\dot{k}_\eta\,\mathrm{d}\eta\right|.
\end{aligned}
$$

Assumption (H2) on σ implies that

$$
\begin{aligned}
|\Delta_{(r_1,r_2)}\Psi(z)| \leqslant{}& L\varphi(r_1r_2)||S^x(h)-S^x(k)||\int_{R_z}|\dot{k}_\eta|\,\mathrm{d}\eta\\
&+L||S^x(h)-S^x(k)||\int_{R_{z,z+r}}|\dot{k}_\eta|\,\mathrm{d}\eta\\
&+L\varphi(r_1)||S^x(h)-S^x(k)||\int_0^s\int_t^{t+r_2}|\dot{k}_\eta|\,\mathrm{d}\eta\\
&+L\varphi(r_2)||S^x(h)-S^x(k)||\int_0^{s+r_1}\int_0^t|\dot{k}_\eta|\,\mathrm{d}\eta
\end{aligned}
$$

since $\Gamma(R_{z,z+r})=r_1r_2$. Using Cauchy–Schwarz inequality we get

$$|\Delta_{(r_1,r_2)}\Psi(z)|\leqslant L||S^x(h)-S^x(k)||\ ||k||_{\mathscr{H}}\ \theta(r_1,r_2,s,t)$$

where

$$\theta(r_1,r_2,s,t)=\varphi(r_1r_2)\sqrt{st}+\sqrt{r_1r_2}+\varphi(r_1)\sqrt{sr_2}+\varphi(r_2)\sqrt{tr_1}.$$

Therefore,

$$
\begin{aligned}
\omega_p(\Psi,(t_1,t_2)) ={}& \sup_{\substack{|r_2|\leqslant t_2\\ |r_1|\leqslant t_1}}||\Delta_{(r_1,r_2)}\Psi||_p\\
&\leqslant L||S^x(h)-S^x(k)||\ ||k||_{\mathscr{H}}\sup_{\substack{|r_2|\leqslant t_2\\ |r_1|\leqslant t_1}}\left(\int_{I_r^2}|\theta(r_1,r_2,s,t)|^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p},
\end{aligned}
$$

where $I_r^2=\{z\in I^2,\ z+r\in I^2\}$. But,

$$
\begin{aligned}
&\left(\int_{I_r^2}|\theta(r_1,r_2,s,t)|^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}\\
&\quad\leqslant 4\varphi(r_1r_2)\left(\int_{I_r^2}\sqrt{st}^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}+4\sqrt{r_1r_2}\\
&\qquad+4\varphi(r_1)\sqrt{r_2}\left(\int_{I_r^2}\sqrt{s}^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}+4\varphi(r_2)\sqrt{r_1}\left(\int_{I_r^2}\sqrt{t}^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}.
\end{aligned}
$$

Since φ is an increasing function we get

$$\sup_{\substack{|r_2|\leqslant t_2\\|r_1|\leqslant t_1}}\left(\int_{I_r^2}|\theta(r_1,r_2,s,t)|^p\,\mathrm{d}s\,\mathrm{d}t\right)^{1/p}\leqslant 4(\varphi(t_1t_2)+\sqrt{t_1t_2}+\varphi(t_1)\sqrt{t_2}+\varphi(t_2)\sqrt{t_1}).$$

Hence,

$$\begin{aligned}\omega_\tau(\Psi,(t_1,t_2)) &= \sup_{p\geqslant 1}\frac{\omega_p(\Psi,(t_1,t_2))}{\sqrt{p}}\\ &\leqslant 4L||S^x(h)-S^x(k)||\;||k||_{\mathscr{H}}(\varphi(t_1t_2)+\sqrt{t_1t_2})\\ &\quad +4L||S^x(h)-S^x(k)||\;||k||_{\mathscr{H}}(\varphi(t_1)\sqrt{t_2}+\varphi(t_2)\sqrt{t_1}),\end{aligned}$$

which implies that

$$\sup_{0<t_1,t_2\leqslant 1}\frac{\omega_\tau(\Psi,(t_1,t_2))}{\omega(t_1t_2)}\leqslant 4L||S^x(h)-S^x(k)||\;||k||_{\mathscr{H}}(D_2+D_3) \tag{16}$$

with

$$D_2:=\sup_{0<t_1,t_2\leqslant 1}\frac{\varphi(t_1t_2)+\sqrt{t_1t_2}}{\omega(t_1)\omega(t_2)}$$

and

$$D_3:=\sup_{0<t_1,t_2\leqslant 1}\frac{\varphi(t_1)\sqrt{t_2}+\varphi(t_2)\sqrt{t_1}}{\omega(t_1)\omega(t_2)}$$

which are finite. Summing up, inequalities (14)–(16) imply that I_2 tends to 0 as $||h-k||_{\tau,\omega}\to 0$.

To complete the proof of the Proposition 9 it remains to prove that I_3 tends to zero when $||h-k||_{\tau,\omega}$ tends to zero.

Define the following approximation of $S^x(h)$

$$S^x(h)^N_\eta = S^x(h)_{(\frac{j}{2^N},\frac{k}{2^N})}\quad \text{if } \eta\in\Delta^N_{j,k}:=\left[\frac{j}{2^N},\frac{j+1}{2^N}\right[\times\left[\frac{k}{2^N},\frac{k+1}{2^N}\right[$$

and

$$\eta^N=\left(\frac{j}{2^N},\frac{k}{2^N}\right)\quad \text{if } \eta\in\Delta^N_{j,k},$$

for all $j=0,\dots,2^N-1$ and $k=0,\dots,2^N-1$. Now,

$$\begin{aligned}I_3 &= \left|\left|\int_{R.}\sigma(\cdot;\eta,S^x(h)_\eta)(\dot h_\eta-\dot k_\eta)\,\mathrm{d}\eta\right|\right|_{\tau,\omega}\\ &\leqslant \left|\left|\int_{R.}(\sigma(\cdot;\eta^N,S^x(h)^N_\eta)-\sigma(\cdot;\eta,S^x(h)_\eta))(\dot h_\eta-\dot k_\eta)\,\mathrm{d}\eta\right|\right|_{\tau,\omega}\\ &\quad +\left|\left|\int_{R.}\sigma(\cdot;\eta^N,S^x(h)^N_\eta)(\dot h_\eta-\dot k_\eta)\,\mathrm{d}\eta\right|\right|_{\tau,\omega}\\ &= I_3^1+I_3^2.\end{aligned}$$

In view of the above calculations, I_3^1 can be estimated as I_2 by replacing $||k||_{\mathscr{H}}$ by $||h-k||_{\mathscr{H}}$ and $S^x(k)_\eta$ by $S^x(h)^N_\eta$ in Eqs. (14)–(16). Hence, with $D=D_1+D_2+D_3$,

$$I_3^1\leqslant DL||S^x(k)-S^x(k)^N||\,||h-k||_{\mathscr{H}}.$$

Now, in Appendix B below we prove that

$$I_3^2 \leqslant 2^{2N}\tilde{M}||h-k||_{\tau,\omega}. \tag{17}$$

Therefore,

$$I_3 \leqslant 2aDL||S^x(h)-S^x(h)^N|| + 2^{2N}\tilde{M}||h-k||_{\tau,\omega}.$$

Combining this inequality and Eqs. (14)–(16) and letting $||h-k||_{\tau,\omega}\to 0$ and then $N\to\infty$ we are done. $\square$

In the proof of Proposition 10, we closely follow the strategy used in Ben Arous and Ledoux (1994) and Eddahbi (1997). In particular, we give an exponential upper bound of the probability that simultaneously the Besov–Orlicz norm of stochastic integrals w.r.t the Brownian sheet is large when the uniform norm of the Brownian sheet is small. This is established in the following series of lemmas:

Lemma 11. *For every $u>0$ and $v>0$ with $u>16v$ and $u>2\sqrt{\log 2}$*

$$P[||W||_* \geqslant u,\ ||W||\leqslant v]\leqslant 6 2^{g^{-1}(u^2/16v^2)}\exp\left(-\frac{u^2}{\log 2}\log\left(\frac{u}{16v}\right)\right),$$

where, g^{-1} is the inverse function of $g(r)=2^r/(1+r)$, $r\geqslant 1$.

Consequently, the inequality holds with the norm $||\cdot||^{**}$ instead of $||\cdot||_*$.

Proof. Let $u>0$ and $v>0$ and recall that

$$||W||_* = \max(|W(1,1)|, ||W||_{*1}, ||W||_{*2}, ||W||_{*3})$$

with

$$||W||_{*1} = \sup_{j\geqslant 0}\ \sup_{2^j+1\leqslant n\leqslant 2^{j+1}} \frac{|C_n(W(.,1))|}{\sqrt{1+j}},$$

$$||W||_{*2} = \sup_{j\geqslant 0}\ \sup_{2^j+1\leqslant n\leqslant 2^{j+1}} \frac{|C_n(W(1,.))|}{\sqrt{1+j}},$$

$$||W||_{*3} = \sup_{j,j'\geqslant 0}\ \sup_{(n,n')\in\mathscr{K}_{j,j'}} \frac{|C_{n,n'}(W)|}{\sqrt{(1+j)(1+j')}}.$$

Here, $\mathscr{K}_{j,j'}=\{(n,n');\ 2^j+1\leqslant n\leqslant 2^{j+1},\ 2^{j'}+1\leqslant n'\leqslant 2^{j'+1}\}$. We have

$$P[||W||_*\geqslant u\ ,\ ||W||\leqslant v]\leqslant P_0+P_1+P_2+P_3,$$

where

$$P_0 = P[|W(1,1)|\geqslant u,\ ||W||\leqslant v],$$

$$P_1 = P[||W||_{*1}\geqslant u,\ ||W||\leqslant v],$$

$$P_2 = P[||W||_{*2}\geqslant u,\ ||W||\leqslant v],$$

$$P_3 = P[||W||_{*3}\geqslant u,\ ||W||\leqslant v].$$

Now, since $u > v$, $P_0 = 0$. We are going to estimate P_1, P_2 and P_3. We have

$$P_1 = P\left[\sup_{j\geqslant 0}\sup_{2^j+1\leqslant n\leqslant 2^{j+1}} \frac{|C_n(W(.,1))|}{\sqrt{1+j}} \geqslant u,\ ||W||\leqslant v\right]$$

$$\leqslant \sum_{j\geqslant 0}\sum_{n=2^j+1}^{2^{j+1}} P[|C_n(W(\cdot,1))|\geqslant u\sqrt{1+j},\ ||W||\leqslant v].$$

Using the fact that, $C_n(W(.,1))$ are one-dimensional centred Gaussian random variables with variance equals to 1, and $|C_n(W(.,1))|$ is dominated by $4v\sqrt{2^j}$ on the set $\{||W||\leqslant v\}$, it follows that, if

$$j_0 = \inf\left\{ j\geqslant 1;\ (1+j)2^{-j} \leqslant \left(\frac{4v}{u}\right)^2 \right\},$$

i.e. we may choose j_0 such that

$$j_0 = \left[g^{-1}\left(\frac{u^2}{16v^2}\right)\right] + 1$$

with g^{-1} being the inverse function of $g(r) = 2^r/(1+r)$, $r\geqslant 1$ and $[r]$ the entire part of the positive number r, we have

$$P_1 \leqslant \sum_{j\geqslant j_0} 2^j P[|N(0,1)|\geqslant u\sqrt{1+j}\,]$$

$$\leqslant 22^{j_0}\exp(-u^2 j_0/2).$$

Here, we used the exponential inequality for a one-dimensional centred Gaussian variable $N(0,1)$:

$$P[|N(0,1)|\geqslant a]\leqslant 2\exp(-a^2/2).$$

Therefore,

$$P_1 \leqslant 22^{g^{-1}(u^2/16v^2)}\exp\left(-\frac{u^2}{\log 2}\log\left(\frac{u}{4v}\right)\right).$$

P_2 can be estimated by the same arguments and therefore we omit the details. Hence,

$$P_1 + P_2 \leqslant 42^{g^{-1}(u^2/16v^2)}\exp\left(-\frac{u^2}{\log 2}\log\left(\frac{u}{4v}\right)\right). \tag{18}$$

It remains to estimate P_3. We have

$$P_3 = P\left[\sup_{j,j'\geqslant 0}\sup_{(n,n')\in\mathscr{K}_{j,j'}} \frac{|C_{n,n'}(W)|}{\sqrt{(1+j)(1+j')}} \geqslant u,\ ||W||\leqslant v\right]$$

$$\leqslant \sum_{j,j'\geqslant 0}\sum_{(n,n')\in\mathscr{K}_{j,j'}} P[|C_{n,n'}(W)|\geqslant u\sqrt{(1+j)(1+j')},\ ||W||\leqslant v].$$

Since $C_{n,n'}(W)$ are one-dimensional centred Gaussian random variables with variance equals to 1, and $|C_{n,n'}(W)|$ is dominated by $16v\sqrt{2^{j'+j}}$ on the set $\{||W||\leqslant v\}$, it follows that

$$\sum_{j,j'\geqslant 0}\ \sum_{(n,n')\in\mathscr{K}_{j,j'}} P[|C_{n,n'}(W)|\geqslant u\sqrt{(1+j)(1+j')},\ ||W||\leqslant v]$$
$$\leqslant \sum_{(j,j');\ j+j'\geqslant k_0} 2^{j'+j}P[|N(0,1)|\geqslant u\sqrt{(1+j)(1+j')}\,],$$

where

$$k_0=\inf\{k\geqslant 1;\ (1+k)2^{-k}\leqslant (16v/u)^2\}.$$

Hence, proceeding as for P_1, we finally get

$$P_3\leqslant 22^{g^{-1}((u/16v)^2)}\exp\left(-\frac{u^2}{\log 2}\log\left(\frac{u}{16v}\right)\right),$$

which, together with (18), give the desired estimate. □

For ease of notation, we denote in the following lemma, $\sigma(z,x)$ for $\sigma(z;\eta,x)$ etc.

Lemma 12. *Let $\tilde{X}^{\varepsilon}$ and Y solve the following stochastic differential equations*:

$$\tilde{X}^{\varepsilon}_z=x_z+\sqrt{\varepsilon}\int_{R_z}\sigma(z,\tilde{X}^{\varepsilon}_{\eta})\tilde{W}(\mathrm{d}\eta)+\int_{R_z}\sigma(z,\tilde{X}^{\varepsilon}_{\eta})\dot{h}_{\eta}\,\mathrm{d}\eta+\int_{R_z}b(z,\tilde{X}^{\varepsilon}_{\eta})\,\mathrm{d}\eta, \tag{19}$$

where $\tilde{W}$ is Brownian sheet and

$$Y_z=x_z+\int_{R_z}\sigma(z,Y_{\eta})\dot{h}_{\eta}\,\mathrm{d}\eta+\int_{R_z}b(z,Y_{\eta})\,\mathrm{d}\eta.$$

Set

$$I^{\varepsilon}_z=\sqrt{\varepsilon}\int_{R_z}\sigma(z,\tilde{X}^{\varepsilon}_{\eta})\tilde{W}(\mathrm{d}\eta).$$

Then there exists a constant $C=C(h,L)>0$ such that

$$||\tilde{X}^{\varepsilon}-Y||^{**}\leqslant C||I^{\varepsilon}||^{**}.$$

Proof. First note that I^{ε} is null on the axes. Thus, expression (8) yields

$$||I^{\varepsilon}||\leqslant ||I^{\varepsilon}||^{**}.$$

On the other hand, using Gronwall's lemma for the two-parameter case, we get

$$||\tilde{X}^{\varepsilon}-Y||\leqslant ||I^{\varepsilon}||\exp\left(\tilde{L}\int_0^1\int_0^1(1+|\dot{h}_{\eta}|\,\mathrm{d}\eta)\right),$$

Hence,

$$||\tilde{X}^{\varepsilon}-Y||\leqslant ||I^{\varepsilon}||^{**}\exp(\tilde{L}(1+||h||_{\mathscr{H}})):=||I^{\varepsilon}||^{**}\tilde{C}. \tag{20}$$

We have

$$||\tilde{X}^{\varepsilon}-Y||^{**}\leqslant ||I^{\varepsilon}||^{**}+\left|\left|\int_{R.}(\phi(\cdot,\tilde{X}^{\varepsilon}_{\eta},Y_{\eta})+\psi(h_{\eta},\cdot,\tilde{X}^{\varepsilon}_{\eta},Y_{\eta}))\,\mathrm{d}\eta\right|\right|^{**}, \tag{21}$$

where

$$\phi(z,x,y)=b(z,x)-b(z,y)$$

and

$$\psi(h_{\cdot}, z, x, y) = (\sigma(z,x) - \sigma(z,y))\dot{h}_{\cdot}\ .$$

Let us first give an estimate of the second term of Eq. (21). From the definition of $||\,.\,||^{**}$ we have

$$\left|\left| \int_{R_{\cdot}} (\phi(\cdot, \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta}) + \psi(h_{\eta}, \cdot, \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta}))\,\mathrm{d}\eta \right|\right|^{**} \leqslant \sum_{r=1}^{6} T_r,$$

where

$$\begin{aligned} T_1 \leqslant &\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\int_{s_1}^{s_2}\int_0^1 |\phi((s_2,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta})|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ &+ \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\int_0^{s_1}\int_0^1 |\phi((s_2,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta}) - \phi((s_1,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta})|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \end{aligned}$$

and

$$\begin{aligned} T_3 \leqslant &\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\int_{s_1}^{s_2}\int_0^1 |\psi(h_{\eta},(s_2,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta})|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ &+ \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\int_0^{s_1}\int_0^1 |\psi(h_{\eta},(s_2,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta}) - \psi(h_{\eta},(s_1,1), \tilde{X}^{\varepsilon}_{\eta}, Y_{\eta})|\,\mathrm{d}\eta}{\omega(s_2-s_1)}. \end{aligned}$$

Furthermore, Assumption (H2) and (20) imply that

$$\begin{aligned} T_1 \leqslant &\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{L\int_{s_1}^{s_2}\int_0^1 |\tilde{X}^{\varepsilon}_{\eta} - Y_{\eta}|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ &+ \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{L\int_0^{s_1}\int_0^1 \varphi(s_2-s_1)|\tilde{X}^{\varepsilon}_{\eta} - Y_{\eta}|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ \leqslant &\ L\tilde{C}||I^{\varepsilon}||^{**} \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\varphi(s_2-s_1)+(s_2-s_1)}{\omega(s_2-s_1)} \end{aligned}$$

and

$$\begin{aligned} T_3 \leqslant &\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{L\int_{s_1}^{s_2}\int_0^1 |\tilde{X}^{\varepsilon}_{\eta} - Y_{\eta}||\dot{h}_{\eta}|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ &+ \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{L\int_0^{s_1}\int_0^1 \varphi(s_2-s_1)|\tilde{X}^{\varepsilon}_{\eta} - Y_{\eta}||\dot{h}_{\eta}|\,\mathrm{d}\eta}{\omega(s_2-s_1)} \\ \leqslant &\ 2L\tilde{C}||I^{\varepsilon}||^{**}(1+||h||_{\mathscr{H}}) \sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\varphi(s_2-s_1)+\sqrt{s_2-s_1}}{\omega(s_2-s_1)}. \end{aligned}$$

Since

$$\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\varphi(s_2-s_1)+(s_2-s_1)}{\omega(s_2-s_1)} :=\tilde{D}_1$$

and

$$\sup_{0\leqslant s_1<s_2\leqslant 1} \frac{\varphi(s_2-s_1)+\sqrt{s_2-s_1}}{\omega(s_2-s_1)} :=\tilde{D}_2$$

are finite we find that

$$T_1+T_3\leqslant 2L\tilde{C}\|I^{\varepsilon}\|^{**}(1+\|h\|_{\mathscr{H}})(\tilde{D}_1+\tilde{D}_2).$$

In a similar way we get

$$T_2+T_4\leqslant 2L\tilde{C}\|I^{\varepsilon}\|^{**}(1+\|h\|_{\mathscr{H}})(\tilde{D}_1+\tilde{D}_2).$$

Since T_5 can be deduced from T_6 by replacing $\dot{h}$ with 1, it suffices to estimate T_6.

Using the notation $z_1=(s_1,t_1)$, $z_2=(s_2,t_2)$ with $z_1<z_2$ and the fact that

$$\bigg|\int_0^{s_2}\int_0^{t_2}\psi(h_\eta,(s_2,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta-\int_0^{s_1}\int_0^{t_2}\psi(h_\eta,(s_1,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta$$
$$-\int_0^{s_2}\int_0^{t_1}\psi(h_\eta,(s_2,t_1),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta+\int_0^{s_1}\int_0^{t_1}\psi(h_\eta,(s_1,t_1),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta\bigg|$$

is dominated by

$$\left|\int_0^{s_1}\int_0^{t_1}\psi(h_\eta,R_{z_1,z_2},\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta\right|+\left|\int_{s_1}^{s_2}\int_{t_1}^{t_2}\psi(h_\eta,(s_2,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)\,\mathrm{d}\eta\right|$$
$$+\left|\int_0^{s_1}\int_{t_1}^{t_2}(\psi(h_\eta,(s_2,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)-\psi(h_\eta,(s_1,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta))\,\mathrm{d}\eta\right|$$
$$+\left|\int_{s_1}^{s_2}\int_0^{t_1}(\psi(h_\eta,(s_2,t_2),\tilde{X}^{\varepsilon}_{\eta},Y_\eta)-\psi(h_\eta,(s_2,t_1),\tilde{X}^{\varepsilon}_{\eta},Y_\eta))\,\mathrm{d}\eta\right|.$$

Assumption (H2) implies that

$$T_6\leqslant\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\varphi((s_2-s_1)(t_2-t_1))}{\omega(s_2-s_1)\omega(t_2-t_1)}L\int_0^{s_1}\int_0^{t_1}|\tilde{X}^{\varepsilon}_{\eta}-Y_\eta||\dot{h}_\eta|\,\mathrm{d}\eta$$
$$+\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{L\int_{s_1}^{s_2}\int_{t_1}^{t_2}|\tilde{X}^{\varepsilon}_{\eta}-Y_\eta||\dot{h}_\eta|\,\mathrm{d}\eta}{\omega(s_2-s_1)\omega(t_2-t_1)}$$
$$+\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\varphi(s_2-s_1)}{\omega(s_2-s_1)\omega(t_2-t_1)}L\int_0^{s_1}\int_{t_1}^{t_2}|\tilde{X}^{\varepsilon}_{\eta}-Y_\eta||\dot{h}_\eta|\,\mathrm{d}\eta$$
$$+\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\varphi(t_2-t_1)}{\omega(s_2-s_1)\omega(t_2-t_1)}L\int_{s_1}^{s_2}\int_0^{t_1}|\tilde{X}^{\varepsilon}_{\eta}-Y_\eta||\dot{h}_\eta|\,\mathrm{d}\eta.$$

Set

$$\tilde{D}_3:=\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\varphi((s_2-s_1)(t_2-t_1))}{\omega(s_2-s_1)\omega(t_2-t_1)},$$

$$\tilde{D}_4:=\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\varphi(s_2-s_1)+\sqrt{t_2-t_1}}{\omega(s_2-s_1)\omega(t_2-t_1)}$$

and

$$\tilde{D}_5:=\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\0\leqslant s_1<s_2\leqslant 1}}\frac{\sqrt{(s_2-s_1)(t_2-t_1)}}{\omega(s_2-s_1)\omega(t_2-t_1)}.$$

Thus,

$$T_6 \leqslant 2L\tilde{C}||I^\varepsilon||^{**}(1+||h||_{\mathscr{H}})(\tilde{D}_3+2\tilde{D}_4+\tilde{D}_5),$$

whence,

$$||\tilde{X}^\varepsilon - Y||^{**} \leqslant C||I^\varepsilon||^{**}.$$

The proof of the lemma is now complete. □

Lemma 13. *For every* $R>0$ *and* $\rho>0$ *large enough, there exist* $\varepsilon_0>0$ *and* $\alpha>0$ *such that*

$$P\left[\left|\left|\sqrt{\varepsilon}\int_{R.}\sigma(\cdot;\eta,X^\varepsilon_\eta)W(\mathrm{d}\eta)\right|\right|^{**} \geqslant \rho, ||\sqrt{\varepsilon}W|| \leqslant \alpha\right] \leqslant \exp -\left(\frac{R}{\varepsilon}\right),$$

for every $\varepsilon\in(0,\varepsilon_0]$.

Proof. For any $N>0$, we consider the approximating sequence of the process $X^\varepsilon_\cdot$ defined by

$$X^{\varepsilon,N}_\eta = X^\varepsilon_{(j/2^N,k/2^N)} \quad \text{if } \eta\in\Delta^N_{jk} = \left[\frac{j}{2^N},\frac{j+1}{2^N}\right[\left[\frac{k}{2^N},\frac{k+1}{2^N}\right[$$

for all $j=0,\dots,2^N-1$ and $k=0,\dots,2^N-1$.

Following similar arguments as in Corollary 3.8 in [RS] and using some exponential inequalities (see also N'zi (1994) or Priouret (1982)), it is easy to check that for all $R>0$ and all $\beta>0$, there exist ε_1 and N_1 such that for every $0<\varepsilon<\varepsilon_1$ and $N\geqslant N_1$,

$$P[||X^\varepsilon - X^{\varepsilon,N}||\geqslant\beta]\leqslant\tfrac{1}{3}\exp(-R/\varepsilon).$$

We have

$$\begin{aligned}
&P\left[\left|\left|\sqrt{\varepsilon}\int_{R.}\sigma(\cdot;\eta,X_\eta)W(\mathrm{d}\eta)\right|\right|^{**}\geqslant\rho,\ ||\sqrt{\varepsilon}W||\leqslant\alpha\right]\\
&\quad\leqslant P[||X^\varepsilon_\cdot - X^{\varepsilon,N}_\cdot||\geqslant\beta]\\
&\qquad + P\left[2\left|\left|\sqrt{\varepsilon}\int_{R.}(\sigma(\cdot;\eta,X^\varepsilon_\eta)-\sigma(\cdot;\eta^N,X^{\varepsilon,N}_\eta))W(\mathrm{d}\eta)\right|\right|^{**}\right.\\
&\qquad\qquad \left.\geqslant\rho,\ ||X^\varepsilon_\cdot - X^{\varepsilon,N}_\cdot||<\beta\right]\\
&\qquad + P\left[2\left|\left|\sqrt{\varepsilon}\int_{R.}\sigma(\cdot;\eta^N,X^{\varepsilon,N}_\eta)W(\mathrm{d}\eta)\right|\right|^{**}\geqslant\rho,\ ||\sqrt{\varepsilon}W||\leqslant\alpha\right].
\end{aligned}$$

By the Lipschitz condition on σ i.e. (H2) and Propositions 6 and 7 the second term in the right-hand side is dominated by

$$K_1\exp\left(-\frac{\rho^2}{4\varepsilon\beta^2K_2}\right),$$

with K_1 and K_2 are some positive constants. Moreover, in view of the arguments used in Appendix B, by replacing $d(h_\eta - k_\eta)$ with $W(\mathrm{d}\eta)$, it follows immediately that

$$\left|\left|\int_{R.}\sigma(\cdot;\eta^N,X^{\varepsilon,N}_\eta)W(\mathrm{d}\eta)\right|\right|^{**}\leqslant 2^{2N}\tilde{M}||W||^{**}.$$

Hence, in view of Lemma 11 we conclude that

$$P\left[\left|\left|\sqrt{\varepsilon}\int_{R.}\sigma(\cdot;\eta^N,X_\eta^{\varepsilon,N})W(\mathrm{d}\eta)\right|\right|^{**}\geqslant\rho,\ ||\sqrt{\varepsilon}W||\leqslant\alpha\right]$$
$$\leqslant 62^{g^{-1}((\rho/(2^{2N}\tilde{M})4\alpha)^2)}\exp\left(-\frac{\rho^2}{\varepsilon(2^{2N}\tilde{M})^2\log 2}\log\left(\frac{\rho}{(2^{2N}\tilde{M})16\alpha}\right)\right).$$

For a given $R>0$ and $\rho>0$ we choose $\beta>0$ small enough so that $\rho^2/4\beta^2K_2>R$, then we choose

$$\varepsilon_2<\frac{1}{\log(3K_1)}\left(\frac{\rho^2}{4\beta^2K_2}-R\right)$$

and then we choose N large enough and η sufficiently small such that

$$62^{g^{-1}((\rho/(2^{2N}\tilde{M})4\alpha)^2)}\exp\left(-\frac{\rho^2}{\varepsilon(2^{2N}\tilde{M})^2\log 2}\log\left(\frac{\rho}{(2^{2N}\tilde{M})16\alpha}\right)\right)\leqslant\frac{1}{3}\exp\left(-\frac{R}{\varepsilon}\right),$$

for $0<\varepsilon\leqslant\varepsilon_0=\min(\varepsilon_1,\varepsilon_2)$, which completes the proof of the lemma. □

Combining Lemmas 12 and 13 we get the following:

Lemma 14. *For every $R>0$, $a>0$ and $\rho>0$ large enough there exist $\alpha>0$ and $\varepsilon_0\in]0,1]$ such that for every $h\in L(a)$ and $\varepsilon\in]0,\varepsilon_0]$ we have*

$$P[||\tilde{X}^\varepsilon-S^x(h)||_{\tau,\omega}>\rho,||\sqrt{\varepsilon}\tilde{W}||\leqslant\alpha]\leqslant\exp-\left(\frac{R}{\varepsilon}\right),$$

where $\tilde{X}^\varepsilon$ satisfies Eq. (19) *and $\tilde{W}:=W-h/\sqrt{\varepsilon}$.*

Proof of Proposition 10. Let $h\in L(a)$ and define on $(\Omega,\mathscr{F},(\mathscr{F}_z)_{z\in I^2})$ the probability Q^ε by

$$\frac{\mathrm{d}Q^\varepsilon}{\mathrm{d}P}:=\exp\left(\frac{1}{\sqrt{\varepsilon}}\int_{I^2}\dot{h}_\eta W(\mathrm{d}\eta)-\frac{\mu(h)}{\varepsilon}\right).$$

Then, under Q^ε, $\tilde{W}:=W-h/\sqrt{\varepsilon}$ is a Wiener process. Now, set

$$A:=\{||X^\varepsilon-S^x(h)||^{**}\geqslant\rho,\ ||\sqrt{\varepsilon}W-h||\leqslant\alpha\}$$

and

$$Z^\varepsilon=\exp\left(-\frac{1}{\sqrt{\varepsilon}}\int_{I^2}\dot{h}_\eta W(\mathrm{d}\eta)\right).$$

Thus,

$$P(A)\leqslant E_{Q^\varepsilon}\left[\frac{\mathrm{d}P}{\mathrm{d}Q^\varepsilon};\ A\cap\left(Z^\varepsilon<\exp\left(\frac{\lambda}{\varepsilon}\right)\right)\right]+P\left(Z^\varepsilon\geqslant\exp\left(\frac{\lambda}{\varepsilon}\right)\right)$$
$$\leqslant\exp\left(\frac{a+\lambda}{\varepsilon}\right)Q^\varepsilon(||\tilde{X}^\varepsilon-S^x(h)||^{**}\geqslant\rho,||\sqrt{\varepsilon}\tilde{W}||\leqslant\alpha)+\exp\left(\frac{a-\lambda}{\varepsilon}\right).$$

Now, choose $\lambda_0=\lambda_0(R,a)$ such that

$$\exp\left(\frac{a-\lambda}{\varepsilon}\right)\leqslant\frac{1}{2}\exp\left(-\frac{R}{\varepsilon}\right)\quad\text{for }\lambda>\lambda_0,$$

then apply Lemma 14 (with Q^ε instead of P) to choose α and ε_0 such that

$$\exp\left(\frac{a+\lambda}{\varepsilon}\right) Q^\varepsilon(\|\tilde{X}^\varepsilon - S^x(h)\|^{**} \geqslant \rho,\ \|\sqrt{\varepsilon}\tilde{W}\| \leqslant \alpha) \leqslant \frac{1}{2}\exp\left(-\frac{R}{\varepsilon}\right)$$

for all $\varepsilon \in (0, \varepsilon_0]$, which ends the proof of the proposition. $\square$

5. For Further Reading

The following reference is also of interest to the reader: Roynette, 1993

Acknowledgements

We would like to thank the two referees of the paper for several insightful comments.

Appendix A

In this appendix we sketch the proofs of Propositions 5–7. In the sequel, we denote

$$\mathscr{K}_{j,j'} = \{(n,n');\ 2^j + 1 \leqslant n \leqslant 2^{j+1},\ 2^{j'} + 1 \leqslant n' \leqslant 2^{j'+1}\}.$$

Proof of Proposition 5. Let $M(s_1,t_1) = M([0,s_1] \times [0,t_1]) = \int_0^{s_1}\int_0^{t_1} H(s,t)W(\mathrm{d}s,\mathrm{d}t)$. We have

$$P[\|M\|_* \geqslant u,\ \|H\| \leqslant 1] \leqslant A_0 + A_1 + A_2 + A_3,$$

where

$$A_0 = P[|M(1,1)| \geqslant u,\ \|H\| \leqslant 1],$$

$$A_1 = P[\|M\|_{*1} \geqslant u,\ \|H\| \leqslant 1],$$

$$A_2 = P[\|M\|_{*2} \geqslant u,\ \|H\| \leqslant 1],$$

$$A_3 = P[\|M\|_{*3} \geqslant u,\ \|H\| \leqslant 1].$$

To estimate A_i for $i = 0,\ldots,3$, we note that, since $M(\cdot,\cdot)$ vanishes on the axes, Proposition 4 yields

$$A_0 \leqslant 4\exp(-u^2/18). \tag{A.1}$$

Moreover,

$$\begin{aligned} A_1 + A_2 \leqslant & \sum_{j\geqslant 0}\sum_{n=2^j+1}^{2^{j+1}} P[|C_n(M(.,1))| \geqslant u\sqrt{1+j},\ \|H\| \leqslant 1] \\ & + \sum_{j\geqslant 0}\sum_{n=2^j+1}^{2^{j+1}} P[|C_n(M(1,.))| \geqslant u\sqrt{1+j},\ \|H\| \leqslant 1]. \end{aligned}$$

Now,

$$P[|C_n(M(.,1))|\geqslant u\sqrt{1+j},\ ||H||\leqslant 1]\leqslant B_1+B_2,$$

$$P[|C_n(M(1,.))|\geqslant u\sqrt{1+j},\ ||H||\leqslant 1]\leqslant B_1'+B_2',$$

where, for $r=1,2$,

$$B_r=P\left[\left|\sqrt{2^j}\int_{(2k-r)/2^{j+1}}^{(2k-r+1)/2^{j+1}}\int_0^1 H(s,t)W(\mathrm{d}s,\mathrm{d}t)\right|\geqslant\frac{u}{2}\sqrt{1+j},\ ||H||\leqslant 1\right],$$

$$B_r'=P\left[\left|\sqrt{2^j}\int_0^1\int_{(2k-r)/2^{j+1}}^{(2k-r+1)/2^{j+1}} H(s,t)W(\mathrm{d}s,\mathrm{d}t)\right|\geqslant\frac{u}{2}\sqrt{1+j},\ ||H||\leqslant 1\right].$$

Hence, in view of Proposition 4, we get

$$B_1+B_2\leqslant 8\exp\left(-\frac{u^2(1+j)}{18}\right)$$

and

$$B_1'+B_2'\leqslant 8\exp\left(-\frac{u^2(1+j)}{18}\right).$$

Therefore,

$$A_1+A_2\leqslant 16\sum_{j\geqslant 0}2^j\exp\left(-\frac{u^2(1+j)}{18}\right).$$

Since $u>12\sqrt{\log 2}$, we get

$$A_1+A_2\leqslant 32\exp\left(-\frac{u^2}{18\times 4}\right). \tag{A.2}$$

To estimate A_3, we remark that, for every $u>12\sqrt{\log 2}$, we have

$$A_3\leqslant 4\sum_{j,j'\geqslant 0}\ \sum_{(n,n')\in\mathscr{K}_{j,j'}} P[|C_{n,n'}(M)|\geqslant u\sqrt{(1+j)(1+j')},\ ||H||\leqslant 1].$$

Using Proposition 4 and the fact that $|C_{n,n'}(M)|$ is dominated by four terms of the form

$$\sqrt{2^{j+j'}}\int_{(k-1)/2^j}^{k/2^j}\int_{(k'-1)/2^{j'}}^{k'/2^{j'}} H(s,t)W(\mathrm{d}s,\mathrm{d}t)$$

whose quadratic variation is bounded by 1, we get

$$\begin{aligned}A_3&\leqslant 4.4\sum_{j,j'\geqslant 0}2^{j'+j}\exp\left(-\frac{u^2(1+j)(1+j')}{4^2\times 18}\right)\\&\leqslant 16\exp\left(-\frac{u^2}{12^2}\right)\left(\sum_{j\geqslant 0}2^j\exp\left(-\frac{ju^2}{2\times 12^2}\right)\right)^2\\&\leqslant 64\exp\left(-\frac{u^2}{12^2}\right),\end{aligned}$$

which together with Eqs. (A.1) and (A.2) yield the estimate of the lemma. $\square$

Proof of Proposition 6. We will only prove (i). The proof of (ii) follows in a similar way.

Set $G(z) = \int_{R_z} H(z,\eta)W(\mathrm{d}\eta)$. Clearly $G(.)$ vanishes on the axes. We have

$$P[||G||_* \geqslant \rho,\ ||H|| \leqslant \alpha] \leqslant A'_0 + A'_1 + A'_2 + A'_3,$$

where

$$A'_0 = P[|G(1,1)| \geqslant \rho,\ ||H|| \leqslant \alpha],$$
$$A'_1 = P[||G||_{*1} \geqslant \rho,\ ||H|| \leqslant \alpha],$$
$$A'_2 = P[||G||_{*2} \geqslant \rho,\ ||H|| \leqslant \alpha],$$
$$A'_3 = P[||G||_{*3} \geqslant \rho,\ ||H|| \leqslant \alpha].$$

We want to estimate A'_i for $i = 0, \ldots, 3$. The exponential inequality in Theorem 2.4 in [RS] yields

$$A'_0 \leqslant 4 \exp -(\rho^2/18C_0\alpha^2), \tag{A.3}$$

for ρ large enough, where C_0 is a numerical constant which can be computed by the Garcia–Rodemich–Rumsey's lemma. Furthermore,

$$A'_1 + A'_2 \leqslant \sum_{j\geqslant 0} \sum_{n=2^j+1}^{2^{j+1}} P[|C_n(G(.,1))| \geqslant \rho\sqrt{1+j},\ ||H|| \leqslant \alpha]$$
$$+ \sum_{j\geqslant 0} \sum_{n=2^j+1}^{2^{j+1}} P[|C_n(G(1,.))| \geqslant \rho\sqrt{1+j},\ ||H|| \leqslant \alpha].$$

Now,

$$P[|C_n(G(.,1))| \geqslant \rho\sqrt{1+j},\ ||H|| \leqslant \alpha] \leqslant B_1 + B_2,$$
$$P[|C_n(G(1,.))| \geqslant \rho\sqrt{1+j},\ ||H|| \leqslant \alpha] \leqslant B'_1 + B'_2,$$

where, for $r = 1, 2$,

$$\begin{aligned}
B_r = {} & P\left[\sqrt{2^j}\left|\int_0^{(2k-r+1)/2^{j+1}} \int_0^1 H\left(\left(\frac{2k-r+1}{2^{j+1}}, 1\right), \eta\right) W(\mathrm{d}\eta)\right.\right. \\
& \left.\left. - \int_0^{(2k-r)/2^{j+1}} \int_0^1 H\left(\left(\frac{2k-r}{2^{j+1}}, 1\right), \eta\right) W(\mathrm{d}\eta)\right| \geqslant \frac{\rho}{2}\sqrt{1+j},\ ||H|| \leqslant \alpha\right] \\
\leqslant {} & P\left[\sqrt{2^j}\left|\int_{(2k-r)/2^{j+1}}^{(2k-r+1)/2^{j+1}} \int_0^1 H\left(\left(\frac{2k-r+1}{2^{j+1}}, 1\right), \eta\right) W(\mathrm{d}\eta)\right|\right. \\
& \left. \geqslant \frac{\rho}{4}\sqrt{1+j},\ ||H|| \leqslant \alpha\right] \\
& + P\left[\sqrt{2^j}\left|\int_0^{(2k-r)/2^{j+1}} \int_0^1 \left(H\left(\left(\frac{2k-r+1}{2^{j+1}}, 1\right), \eta\right)\right.\right.\right. \\
& \left.\left. \left. - H\left(\left(\frac{2k-r}{2^{j+1}}, 1\right), \eta\right)\right) W(\mathrm{d}\eta)\right|\right. \\
& \left. \geqslant \frac{\rho}{4}\sqrt{1+j},\ ||H|| \leqslant \alpha\right],
\end{aligned}$$

and

$$B'_r = P\left[\sqrt{2^j}\left|\int_0^1\int_0^{(2k-r+1)/2^{j+1}} H\left(\left(1,\frac{2k-r+1}{2^{j+1}}\right),\eta\right)W(\mathrm{d}\eta)\right.\right.$$
$$\left.\left.-\int_0^1\int_0^{(2k-r)/2^{j+1}} H\left(\left(1,\frac{2k-r}{2^{j+1}}\right),\eta\right)W(\mathrm{d}\eta))\right|\geqslant\frac{\rho}{2}\sqrt{1+j},\ ||H||\leqslant\alpha\right]$$
$$\leqslant P\left[\sqrt{2^j}\left|\int_0^1\int_{(2k-r)/2^{j+1}}^{(2k-r+1)/2^{j+1}} H\left(\left(1,\frac{2k-r+1}{2^{j+1}}\right),\eta\right)W(\mathrm{d}\eta)\right|\right.$$
$$\left.\geqslant\frac{\rho}{4}\sqrt{1+j},\ ||H||\leqslant\alpha\right]$$
$$+P\left[\sqrt{2^j}\left|\int_0^1\int_0^{(2k-r)/2^{j+1}}\left(H\left(\left(1,\frac{2k-r+1}{2^{j+1}}\right),\eta\right)\right.\right.\right.$$
$$\left.\left.-H\left(\left(1,\frac{2k-r}{2^{j+1}}\right),\eta\right)\right)W(\mathrm{d}\eta)\right|$$
$$\left.\geqslant\frac{\rho}{4}\sqrt{1+j},\ ||H||\leqslant\alpha\right].$$

Hence, in view of the exponential inequality of Theorem 2.4 in [RS], we get

$$B_1+B_2\leqslant 16\exp-\left(\frac{\rho^2(1+j)}{72(\alpha^2+L^2)C_0}\right)$$

and

$$B'_1+B'_2\leqslant 16\exp-\left(\frac{\rho^2(1+j)}{72(\alpha^2+L^2)C_0}\right).$$

Therefore,

$$A'_1+A'_2\leqslant 32\sum_{j\geqslant 0}2^j\exp-\left(\frac{\rho^2(1+j)}{72(\alpha^2+L^2)C_0}\right).$$

Since $\rho>6\sqrt{\alpha^2+L^2}\sqrt{2C_0\log 2}$, we get

$$A'_1+A'_2\leqslant 64\exp-\left(\frac{\rho^2}{72(\alpha^2+L^2)C_0}\right). \tag{A.4}$$

To estimate A'_3, we remark that for every ρ sufficiently large,

$$A'_3\leqslant 4\sum_{j,j'\geqslant 0}\sum_{(n,n')\in K_{j,j'}}P[|C_{n,n'}(G)|\geqslant\rho\sqrt{(1+j)(1+j')},\ ||H||\leqslant\alpha]$$

and that $|C_{n,n'}(G)|$ is dominated by four terms of the form

$$\sqrt{2^{j+j'}}(Z_1+Z_2+Z_3+Z_4),$$

where

$$Z_1 = \left| \int_0^{(k-1)/2^j} \int_0^{(k'-1)/2^{j'}} H(R_{((k-1)/2^j,k/2^j),((k'-1)/2^{j'},k'/2^{j'})},\eta) W(\mathrm{d}\eta) \right|,$$

$$Z_2 = \left| \int_0^{(k-1)/2^j} \int_{(k'-1)/2^{j'}}^{k'/2^{j'}} \left(H\left(\left(\frac{k}{2^j},\frac{k'}{2^{j'}}\right),\eta\right) - H\left(\left(\frac{(k-1)}{2^j},\frac{k'}{2^{j'}}\right),\eta\right)\right) W(\mathrm{d}\eta) \right|,$$

$$Z_3 = \left| \int_{(k-1)/2^j}^{k/2^j} \int_{(k'-1)/2^{j'}}^{k'/2^{j'}} H\left(\left(\frac{k}{2^j},\frac{k'}{2^{j'}}\right),\eta\right) W(\mathrm{d}\eta) \right|,$$

$$Z_4 = \left| \int_{(k-1)/2^j}^{k/2^j} \int_0^{(k'-1)/2^{j'}} \left(H\left(\left(\frac{k}{2^j},\frac{k'}{2^{j'}}\right),\eta\right) - H\left(\left(\frac{k}{2^j},\frac{k'-1}{2^{j'}}\right),\eta\right)\right) W(\mathrm{d}\eta) \right|.$$

Since the quadratic variation of each of Z_i's for $i = 1, 2, 4$ is bounded by $L^2 2^{-(j+j')}$ and the quadratic variation of Z_3 is bounded by $\alpha^2 2^{-(j+j')}$, the quadratic variation of $C_{n,n'}(G)$ is dominated by $4(4\alpha^2 + 12L^2)$, the exponential inequality for the uniform norm (Theorem 2.4 in [RS]) yields

$$A_3' \leqslant 4.16 \sum_{j,j' \geqslant 0} 2^{j'+j} \exp - \left(\frac{\rho^2(1+j)(1+j')}{18(4\alpha^2 + 12L^2)C_0} \right)$$

$$\leqslant 64 \exp - \left(\frac{\rho^2}{18(4\alpha^2 + 12L^2)C_0} \right) \left(\sum_{j \geqslant 0} 2^j \exp\left(- \frac{j\rho^2}{18(4\alpha^2 + 12L^2)C_0} \right) \right)^2.$$

Therefore, for $\rho > 6\sqrt{\alpha^2 + 3L^2}\sqrt{6C_0 \log 2}$

$$A_3' \leqslant 4.64 \exp - \left(\frac{\rho^2}{72(\alpha^2 + 3L^2)C_0} \right),$$

which together with Eqs. (A.3) and (A.4) yield the estimate of (i). □

Proof of Proposition 7. We will only prove (i). The proof of (ii) follows using the same arguments. Set $M(s,t) = \int_0^s \int_{R_{s,t}} H(\eta, r) W(\mathrm{d}\eta)\,\mathrm{d}r$. We have

$$P[||M(\cdot,\cdot)||^{**} > \rho,\ ||H|| \leqslant \alpha]$$

$$\leqslant P\left[\sup_{0 \leqslant s_1 < s_2 \leqslant 1} \frac{|M(s_2,1) - M(s_1,1)|}{\omega(s_2 - s_1)} > \frac{\rho}{3}, ||H|| \leqslant \alpha \right]$$

$$+ P\left[\sup_{0 \leqslant t_1 < t_2 \leqslant 1} \frac{|M(1,t_2) - M(1,t_1)|}{\omega(t_2 - t_1)} > \frac{\rho}{3},\ ||H|| \leqslant \alpha \right]$$

$$+ P\left[\sup_{\substack{0 \leqslant t_1 < t_2 \leqslant 1 \\ 0 \leqslant s_1 < s_2 \leqslant 1}} \frac{|M(R_{z_1,z_2})|}{\omega(s_2 - s_1)\omega(t_2 - t_1)} > \frac{\rho}{3}, ||H|| \leqslant \alpha \right]$$

$$= P_1 + P_2 + P_3$$

where

$$M(R_{z_1,z_2}) = M(s_2,t_2) - M(s_2,t_1) - M(s_1,t_2) + M(s_1,t_1).$$

Let us first estimate P_1. We have

$$\begin{aligned}
&|M(s_2,1)-M(s_1,1)|\\
&\quad\leqslant \left|\int_{s_1}^{s_2}\int_{R_{s_2,1}} H(\eta,r)W(\mathrm{d}\eta)\mathrm{d}r\right| + \left|\int_{s_1}^{s_2}\int_0^1\left(\int_0^{s_1} H(\eta,r)\mathrm{d}r\right)W(\mathrm{d}\eta)\right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_1 &\leqslant P\left[\sup_{0\leqslant s_1<s_2\leqslant 1}\frac{|\int_{s_1}^{s_2}\int_0^1(\int_0^{s_1} H(\eta,r)\mathrm{d}r)W(\mathrm{d}\eta)|}{\omega(s_2-s_1)} > \frac{\rho}{6},\ ||H||\leqslant\alpha\right]\\
&\quad + P\left[\sup_{0\leqslant s_1<s_2\leqslant 1}\frac{|\int_{s_1}^{s_2}(\int_{R_{s_2,1}} H(\eta,r)\mathrm{d}r)W(\mathrm{d}\eta))\mathrm{d}r|}{\omega(s_2-s_1)} > \frac{\rho}{6},\ ||H||\leqslant\alpha\right]\\
&= P_1^1 + P_1^2.
\end{aligned}$$

Using Proposition 5, we get

$$P_1^1 \leqslant 100\exp -\left(\frac{\rho^2}{36\cdot 144\alpha^2}\right).$$

Now,

$$\begin{aligned}
P_1^2 &\leqslant P\left[\sup_{0\leqslant r,s_2\leqslant 1}\left|\int_{R_{s_2,1}} H(\eta,r)W(\mathrm{d}\eta)\right| \geqslant \lambda, ||H||\leqslant\alpha\right]\\
&\quad + P\left[\sup_{0\leqslant s_1<s_2\leqslant 1}\frac{|\int_{s_1}^{s_2}(\int_{R_{s_2,1}} H(\eta,r)W(\mathrm{d}\eta))\mathrm{d}r|}{\omega(s_2-s_1)} \geqslant \frac{\rho}{6},\right.\\
&\qquad\left.\sup_{0\leqslant r,s_2\leqslant 1}\left|\int_{R_{s_2,1}} H(\eta,r)W(\mathrm{d}\eta)\right| \leqslant \lambda\right].
\end{aligned}$$

Choosing $3\lambda < \rho$ and applying Proposition 5, we get

$$\begin{aligned}
P_1^2 &\leqslant P\left[\sup_{0\leqslant r,s_2\leqslant 1}\left|\int_{R_{s_2,1}} H(\eta,r)W(\mathrm{d}\eta)\right| \geqslant \lambda, ||H||\leqslant\alpha\right]\\
&\leqslant \exp\left(-\frac{\lambda^2 C_2}{4\alpha^2+12L^2}\right).
\end{aligned}$$

P_2 is estimated in a similar way as P_1. It remains to estimate P_3. We note that

$$\begin{aligned}
|M(R_{z_1,z_2})| &= \left|\int_0^{s_2}\left(\int_{R_{s_2,t_2}} H(\eta,r)W(\mathrm{d}\eta)\mathrm{d}r - \int_{R_{s_2,t_1}} H(\eta,r)W(\mathrm{d}\eta)\right)\mathrm{d}r\right.\\
&\qquad\left. + \int_0^{s_1}\left(\int_{R_{s_1,t_1}} H(\eta,r)W(\mathrm{d}\eta)\mathrm{d}r - \int_{R_{s_1,t_2}} H(\eta,r)W(\mathrm{d}\eta)\right)\mathrm{d}r\right|\\
&\leqslant \left|\int_{s_1}^{s_2}\left(\int_{R_{s_2,t_2}} H(\eta,r)W(\mathrm{d}\eta) - \int_{R_{s_2,t_1}} H(\eta,r)W(\mathrm{d}\eta)\right)\mathrm{d}r\right|\\
&\qquad + \left|\int_0^{s_1}\int_{R_{z_1,z_2}} H(\eta,r)W(\mathrm{d}\eta)\mathrm{d}r\right|.
\end{aligned}$$

Thus,

$$
\begin{aligned}
P_3 &\leqslant P\left[\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\ 0\leqslant s_1<s_2\leqslant 1}} \frac{|\int_{s_1}^{s_2}(\int_0^{s_2}\int_{t_1}^{t_2} H(\eta,r)W(\mathrm{d}\eta))\,\mathrm{d}r|}{\omega(s_2-s_1)\omega(t_2-t_1)} > \frac{\rho}{6},\ ||H||\leqslant\alpha\right]\\
&\quad + P\left[\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\ 0\leqslant s_1<s_2\leqslant 1}} \frac{|\int_{R_{z_1,z_2}}(\int_0^{s_1} H(\eta,r)\mathrm{d}r)W(\mathrm{d}\eta))|}{\omega(s_2-s_1)\omega(t_2-t_1)} > \frac{\rho}{6},\ ||H||\leqslant\alpha\right]\\
&= P_3^1+P_3^2.
\end{aligned}
$$

But, for $6\lambda<\rho$,

$$
\begin{aligned}
P_3^1 &\leqslant P\left[\sup_{\substack{0\leqslant t_1<t_2\leqslant 1\\ 0\leqslant r,s_2\leqslant 1}} \frac{|\int_0^{s_2}\int_{t_1}^{t_2} H(\eta,r)W(\mathrm{d}\eta)|}{\omega(t_2-t_1)} > \lambda, ||H||\leqslant\alpha\right]\\
&\leqslant \exp\left(-\frac{\lambda^2 C_2}{4\alpha^2+12L^2}\right),
\end{aligned}
$$

by Proposition 6. Furthermore on the set $\{||H||\leqslant\alpha\}, ||\int_0^{\cdot} H(\cdot,r)\mathrm{d}r||\leqslant\alpha$. Hence, by Proposition 6, we get

$$
P_3^2\leqslant\exp\left(-\frac{\rho^2 C_2}{4\alpha^2+12L^2}\right).
$$

for ρ sufficiently large. □

Appendix B

In this section we shall prove inequality (17) above. The proof of the inequality

$$
\left|\left|\int_{R.}\sigma(\cdot;\eta^N,X_\eta^{\varepsilon,N})W(\mathrm{d}\eta)\right|\right|^{**}\leqslant 2^{2N}\tilde{M}||W||^{**}
$$

that has been used in the proof of Lemma 13 is omitted, since it can proved using similar arguments for the appropriate norm, by replacing $\mathrm{d}(h_\eta-k_\eta)$ with $W(\mathrm{d}\eta)$.

Set

$$
\phi(z)=\int_{R_z}\sigma(z,\eta^N,S^x(h)_\eta^N)\mathrm{d}(h_\eta-k_\eta),
$$

where

$$
S^x(h)_\eta^N=S^x(h)_{(j/2^N,k/2^N)}\quad\text{if }\eta\in\Delta_{jk}^N=\left[\frac{j}{2^N},\frac{j+1}{2^N}\right[\times\left[\frac{k}{2^N},\frac{k+1}{2^N}\right[
$$

and

$$
\eta^N=\left(\frac{j}{2^N},\frac{k}{2^N}\right)\quad\text{if }\eta\in\Delta_{jk}^N.
$$

In the rest of the proof we will sometimes use the following notation:

$$
f(A)=\int_A\dot{f}_\eta\,\mathrm{d}\eta\quad\text{for }A\in I^2.
$$

We shall prove that there exists a positive constant $\tilde{M}$ such that

$$||\phi||_{\tau,\omega} \leqslant 2^{2N}\tilde{M}||h-k||_{\tau,\omega}.$$

For $r>0$ and $z=(s,t)$, we have

$$|\phi(z+re_1)-\phi(z)| \leqslant \left|\int_s^{s+r}\int_0^t \sigma((s+r,t),\eta^N,S^x(h)_\eta^N)\mathrm{d}(h_\eta-k_\eta)\right| + \left|\int_0^s\int_0^t (\sigma((s+r,t),\eta^N,S^x(h)_\eta^N)-\sigma((s,t),\eta^N,S^x(h)_\eta^N))\mathrm{d}(h_\eta-k_\eta)\right|.$$

Approximating the integrals by Riemann sums, we get

$$\begin{aligned}|\phi(z+re_1)-\phi(z)| \leqslant{}& \left|\sum_{j,l=0}^{2^N-1}\sigma\left((s+r,t),\left(\frac{j}{2^N},\frac{l}{2^N}\right),S^x(h)_{(j/2^N,k/2^N)}\right)\right.\\ &\left.\times(h-k)([s,s+r]\times[0,t]\cap\Delta_{jl}^N)\right|\\ &+\left|\sum_{j,l=0}^{2^N-1}\left(\sigma\left((s+r,t),\left(\frac{j}{2^N},\frac{l}{2^N}\right),S^x(h)_{(j/2^N,k/2^N)}\right)\right.\right.\\ &-\left(\sigma\left((s,t),\left(\frac{j}{2^N},\frac{l}{2^N}\right),S^x(h)_{(j/2^N,k/2^N)}\right)\right)\Bigg)\\ &\left.\times(h-k)([s,s+r]\times[0,t]\cap\Delta_{jl}^N)\right|.\end{aligned}$$

Now, relation (H2) yields

$$\begin{aligned}|\phi(z+re_1)-\phi(z)| \leqslant{}& \sum_{j,l=0}^{2^N-1} M|(h-k)([s,s+r]\times[0,t]\cap\Delta_{jl}^N)|\\ &+\sum_{j,l=0}^{2^N-1} L\varphi(r)|(h-k)([0,s]\times[0,t]\cap\Delta_{jl}^N)|\\ \leqslant{}& 2^{2N}M|(h-k)(z+re_1)-(h-k)(z)|+2^{2N}L\varphi(r)||h-k||.\end{aligned}$$

Similarly,

$$|\phi(z+re_2)-\phi(z)| \leqslant 2^{2N}M|(h-k)(z+re_2)-(h-k)(z)|+2^{2N}L\varphi(r)||h-k||.$$

Hence, for $i=1,2$,

$$||\Delta_{r,i}\phi||_p \leqslant 2^{2N+1-1/p}(M||\Delta_{r,i}(h-k)||_p+L\varphi(r)||h-k||).$$

Therefore,

$$\begin{aligned}\omega_{p,i}(\phi,t) ={}& \sup_{|r|\leqslant t}||\Delta_{r,i}\phi||_p\\ \leqslant{}& 2^{2N+1-1/p}\left(M\sup_{|r|\leqslant t}||\Delta_{r,i}(h-k)||_p+L\varphi(t)||h-k||\right).\end{aligned}$$

and then

$$\omega_{\tau,i}(\phi,t)=\sup_{p\geqslant 1}\frac{\omega_{p,i}(\phi,t)}{\sqrt{p}}\leqslant 2^{2N}(M\omega_{\tau,i}(h-k,t)+L\varphi(t)||h-k||),$$

which implies that

$$\sup_{0<t\leqslant 1}\frac{\omega_{\tau,i}(\phi,t)}{\omega(t)}\leqslant 2^{2N}\left(M\sup_{0<t\leqslant 1}\frac{\omega_{\tau,i}(h-k,t)}{\omega(t)}+L\sup_{0<t\leqslant 1}\frac{\varphi(t)}{\omega(t)}||h-k||\right). \tag{B.1}$$

Now, for $r=(r_1,r_2)\in\mathbb{R}_+^2$, we have

$$\begin{aligned}
&|\varDelta_{(r_1,r_2)}(\phi(z))|\\
&\quad\leqslant\left|\int_{R_z}\sigma(R_{z,z+r},\eta^N,S^x(h)b_\eta^N)\mathrm{d}(h_\eta-k_\eta)\right|\\
&\qquad+\left|\int_{R_{z,z+r}}\sigma(z+r,\eta^N,S^x(h)_\eta^N)\mathrm{d}(h_\eta-k_\eta)\right|\\
&\qquad+\left|\int_0^s\int_t^{t+r_2}(\sigma(z+r,\eta^N,S^x(h)_\eta^N)-\sigma((s,t+r_2),\eta^N,S^x(h)_\eta^N))\mathrm{d}(h_\eta-k_\eta)\right|\\
&\qquad+\left|\int_s^{s+r_1}\int_0^t(\sigma(z+r,\eta^N,S^x(h)_\eta^N)-\sigma((s+r_1,t),\eta^N,S^x(h)_\eta^N))\mathrm{d}(h_\eta-k_\eta)\right|.
\end{aligned}$$

Approximating the terms appearing in the right-hand side by the Riemann sum as above, we get that

$$\begin{aligned}
|\varDelta_{(r_1,r_2)}(\phi(z))|\leqslant{}&2^{2N}(4L\varphi(r_1r_2)||h-k||+M|\varDelta_{(r_1,r_2)}(h-k)(z)|)\\
&+2^{2N}(L\varphi(r_1)|\varDelta_{(r_2,2)}(h-k)(z)|+L\varphi(r_2)|\varDelta_{(r_1,1)}(h-k)(z)|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2^{-2(N+1-1/p)}\omega_p(\phi,(t_1,t_2))\\
&\quad=2^{-2(N+1-1/p)}\sup_{|r_1|\leqslant t_1,|r_2|\leqslant t_2}||\varDelta_{(r_1,r_2)}\phi||_p\\
&\quad\leqslant 4L\varphi(t_1t_2)||h-k||+M\sup_{|r_1|\leqslant t_1,|r_2|\leqslant t_2}||\varDelta_{(r_1,r_2)}(h-k)||_p\\
&\qquad+L\varphi(t_1)\sup_{|r_2|\leqslant t_2}||\varDelta_{(r_2,2)}(h-k)||_p+L\varphi(t_2)\sup_{|r_1|\leqslant t_1}||\varDelta_{(r_1,1)}(h-k)||_p,
\end{aligned}$$

where we have used the assumption that $\varphi(\cdot)$ is increasing. Hence,

$$\omega_\tau(\phi,(t_1,t_2))=\sup_{p\geqslant 1}\frac{\omega_p(\phi,(t_1,t_2))}{\sqrt{p}}=\sup_{p\geqslant 1}\sup_{|r_1|\leqslant t_1,|r_2|\leqslant t_2}\frac{||\varDelta_{(r_1,r_2)}(\phi)||_p}{\sqrt{p}}$$

satisfies

$$\begin{aligned}
&2^{-2N}\sup_{0<t_1,t_2\leqslant 1}\frac{\omega_\tau(\phi,(t_1,t_2))}{\omega(t_1)\omega(t_2)}\\
&\quad\leqslant M\sup_{0<t_1,t_2\leqslant 1}\frac{\omega_\tau(h-k,(t_1,t_2))}{\omega(t_1)\omega(t_2)}+4L||h-k||\sup_{0<t_1,t_2\leqslant 1}\frac{\varphi(t_1t_2)}{\omega(t_1)\omega(t_2)}\\
&\qquad+L\sup_{0<t_1,t_2\leqslant 1}\frac{\varphi(t_1)}{\omega(t_1)}\frac{\omega_{\tau,2}(h-k,t_2)}{\omega(t_2)}+L\frac{\varphi(t_2)}{\omega(t_2)}\frac{\omega_{\tau,1}(h-k,t_1)}{\omega(t_1)}.
\end{aligned}\tag{B.2}$$

Now, combining Eqs. (B.1) and (B.2), together with the definition of the norm $||\cdot||_{\tau,\omega}$, we obtain

$$||\phi||_{\tau,\omega} \leqslant 2^{2N}\tilde{M}||h-k||_{\tau,\omega}.$$

as is claimed. □

References

Azencott, R., 1980. Grandes déviations et applications. Ecole d'Eté de Probabilités de Saint-Flour VIII, 1978. Lecture Notes in Mathematics, vol. 774. Springer, New York, pp. 1–176.

Ben Arous, G., Ledoux, M., 1994. Grandes déviations de Freidlin-Wentzell en norme hölderienne. Séminaire de Probabilités 1583, 293–299.

Ciesielski, Z., 1993. Orlicz spaces, spline systems and Brownian motion. Const. Approx. 9, 191–208.

Ciesielski, Z., Domsta, J., 1972. Construction of an orthonormal basis in $\mathscr{C}^m(I^d)$ and $\mathscr{W}_p^m(I^d)$. Const. Approx. 41, 211–224.

Ciesielski, Z., Kamont, A., 1995. Levy's fractional Brownian random field and function spaces. Acta Sci. Math. (Szeged) 60, 99–118.

Ciesielski, Z., Kerkyacharian, G., Roynette, B., 1993. Quelques espaces fonctionnels associés à des processus gaussiens. Stud. Math. 107, 171–204.

Doss, H., Dozzi, M., 1987. Estimations de grandes déviations pour les processus de diffusion à paramètre multidimensionnel. Lecture Notes in Mathematics, vol. 1204. Springer, Berlin, pp. 68–90.

Dozzi, M., 1989. Stochastic Processes with a Multidimensional Parameter. Longman Scientific & Technical, Essex.

Eddahbi, M., 1997. Large deviations for solutions of hyperbolic SPDEs in Hölder norm. Potential Anal. 7, 517–537.

Fernique, X., 1971. Régularité de processus gaussiens. Invent. Math. 12, 304–320.

Jonsson, A., Wallin, H., 1984. Function spaces on subsets of $\mathbb{R}^n$. Mathematical Reports, vol. 2, Part 1, pp. 1–221. (Issue Ed.), Peetre, J.

Kamont, A., 1994. Isomorphism of some anisotropic Besov and sequence spaces. Stud. Math. 110 (2), 169–189.

Kamont, A., 1996. On the fractional anisotopic Wiener field. Probab. Math. Statist. 16 (1), 85–98.

Mishura, S., 1987. Exponential estimates for two-parameter martingales. Ukrainian Math. J. 39, 275–279.

Norris, J., 1995. Twisted sheets. J. Funct. Anal. 123, 273–334.

N'zi, M., 1994. On the large deviation estimates for two-parameter diffusion processes and applications. Stochastics Stochastics Rep. 50, 65–83.

Peetre, J., 1976. New Thoughts on Besov spaces. Duke Univ. Math. Series, Durham, NC.

Priouret, P., 1982. Remarques sur les petites perturbations de systèmes dynamiques. Séminaire de Probabilités XVI. Lecture Notes in Mathematics, vol. 920. Springer, New York, pp. 184–200.

Rao, M.M., Ren, Z.D., 1991. Theory of Orlicz Spaces. Marcel Dekker, New York.

Ropela, S., 1976. Spline Basis in Besov spaces. Bul Acad. Polon. Scio. Sér. Sci. Math., Astronom. Phys. 24, 319–325.

Roynette, B., 1993. Mouvement brownien et espaces de Besov. Stochastics Stochastics Rep. 43, 221–260.

Rovira, C., Sanz-Solé, M., 1996. The law of the solutions to a nonlinear hyperbolic SPDE. J. Theory Probab. 9 (4), 863–901.

Rovira, C., Sanz-Solé, M., 1997. Large deviations for stochastic Volterra equations in the plane, preprint.